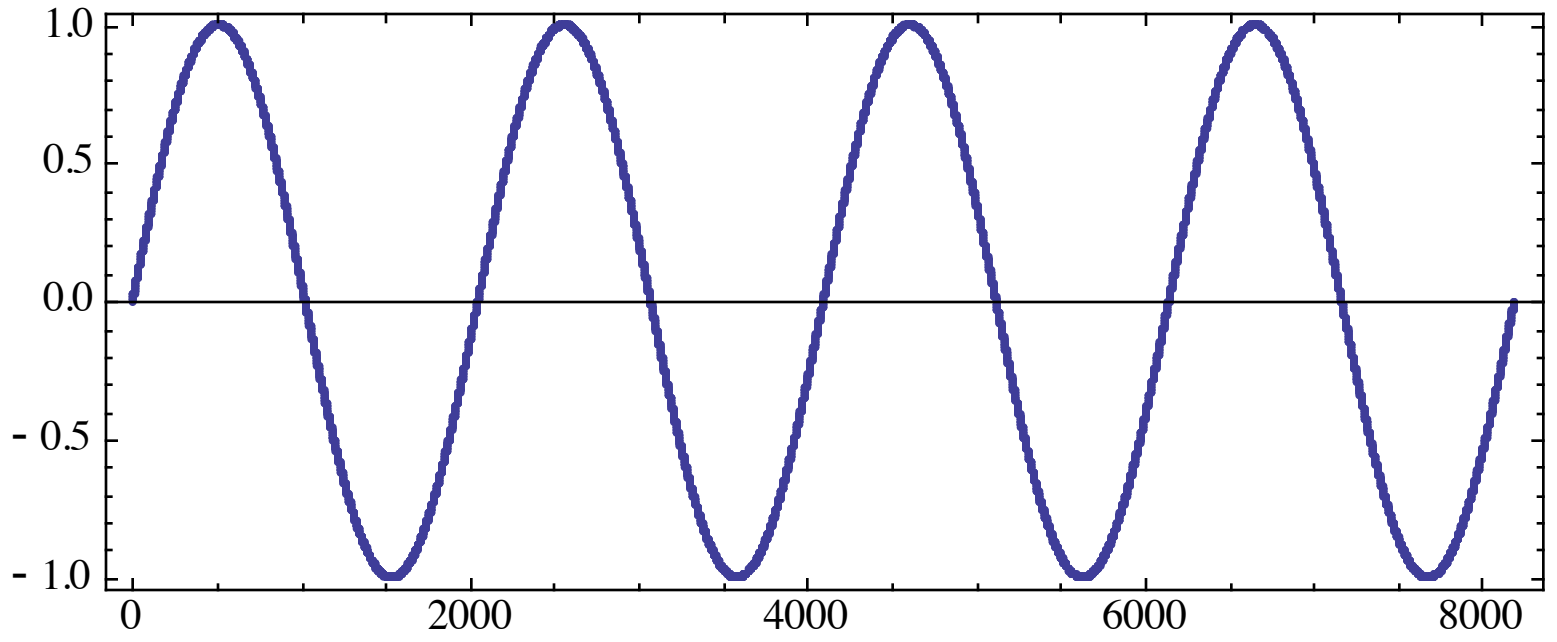
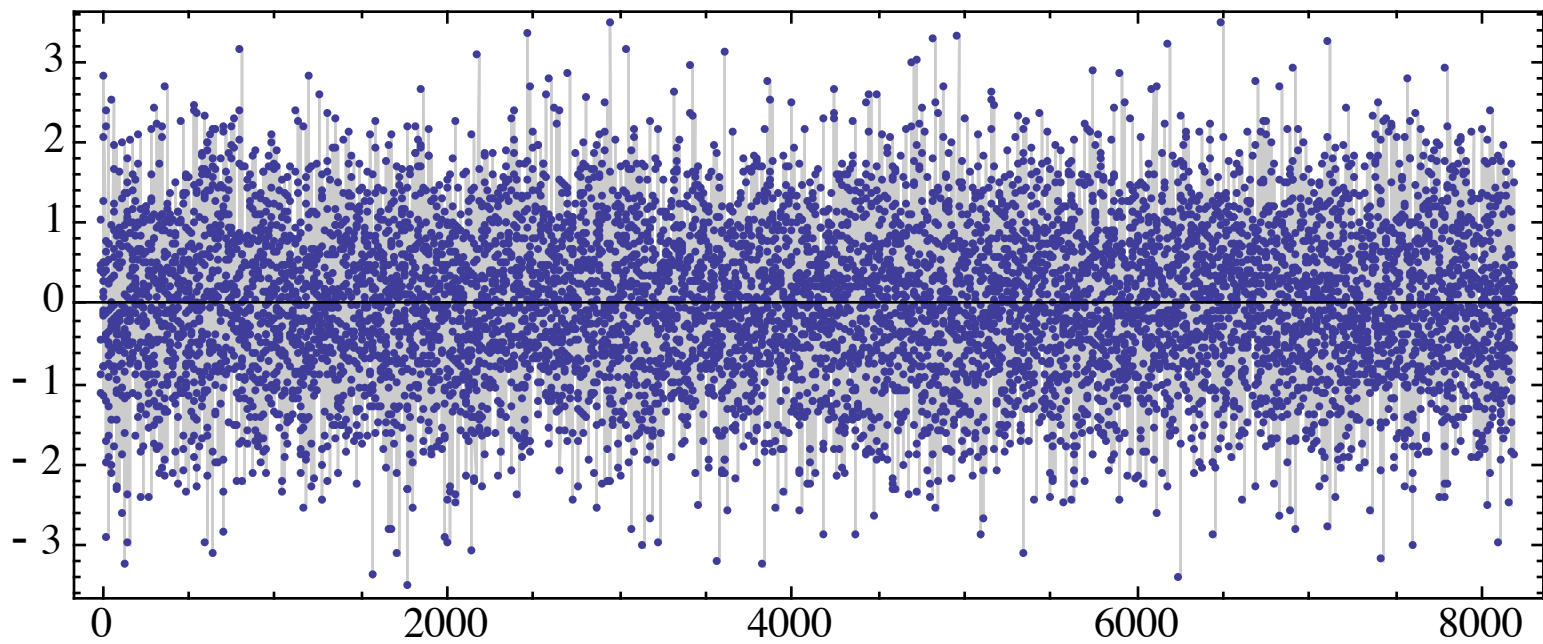


Introduzione al rumore

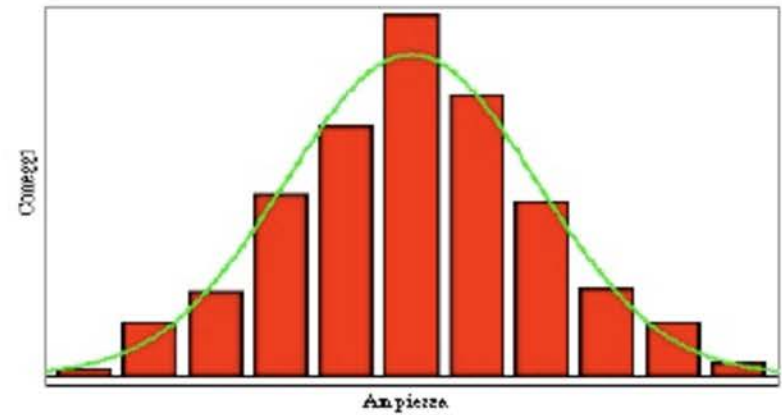
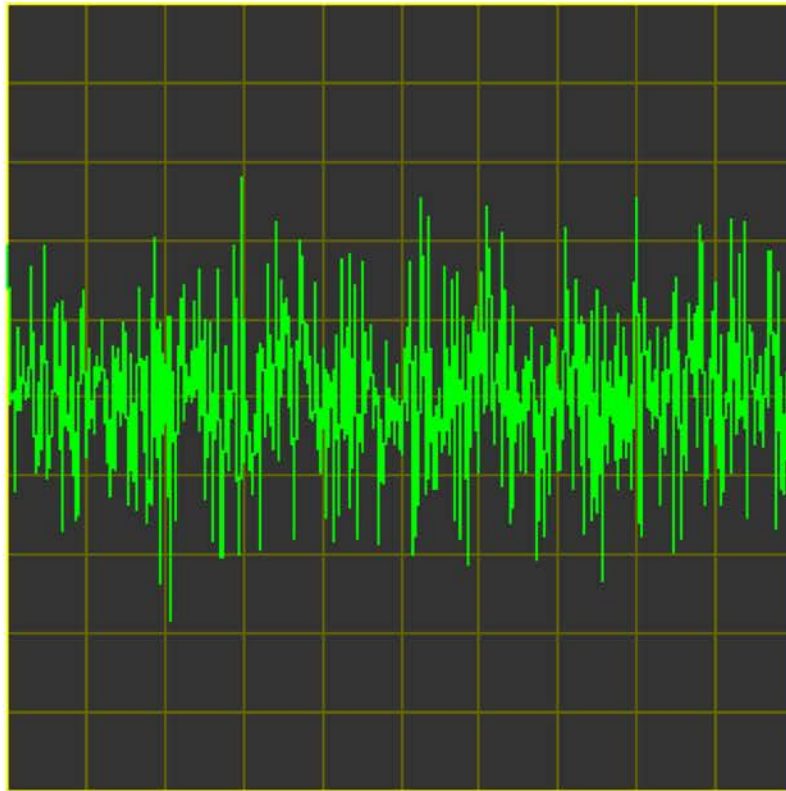
Edoardo Milotti

Corso di Fondamenti Fisici di Tecnologia Moderna



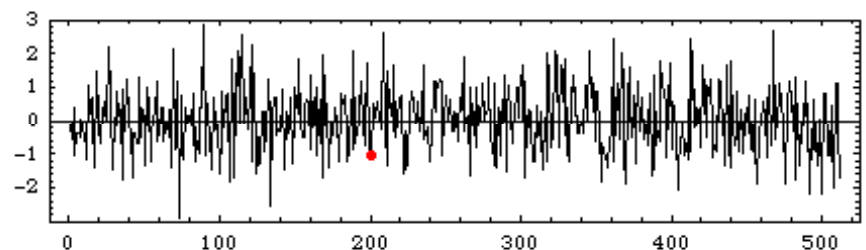
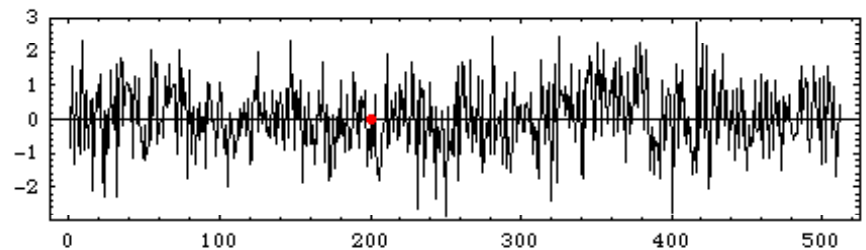
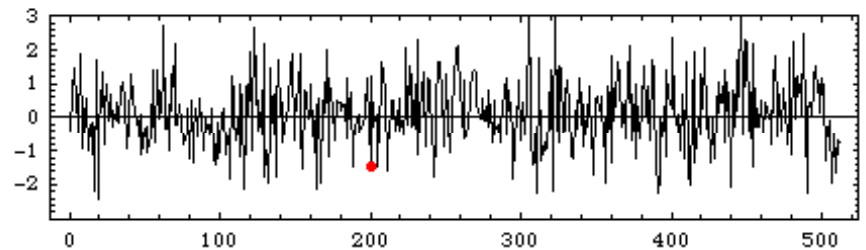
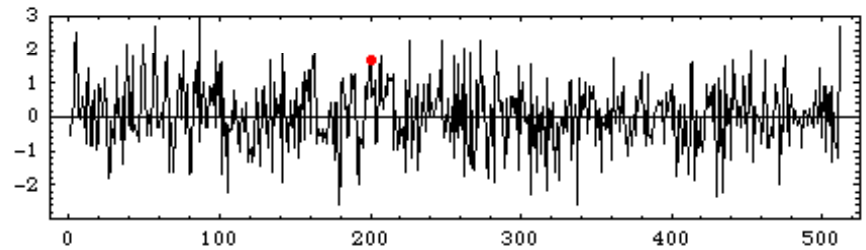
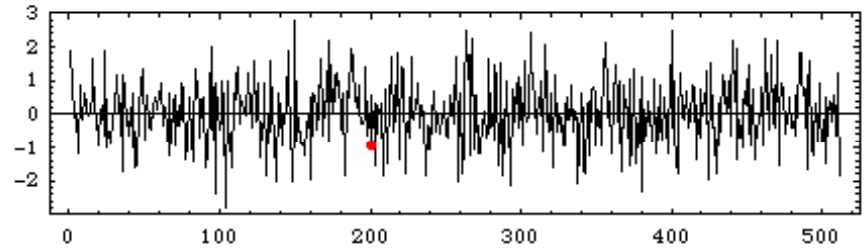


Rumore bianco Gaussiano: campioni scorrelati uno dall'altro, statistica dell' ampiezza Gaussiana

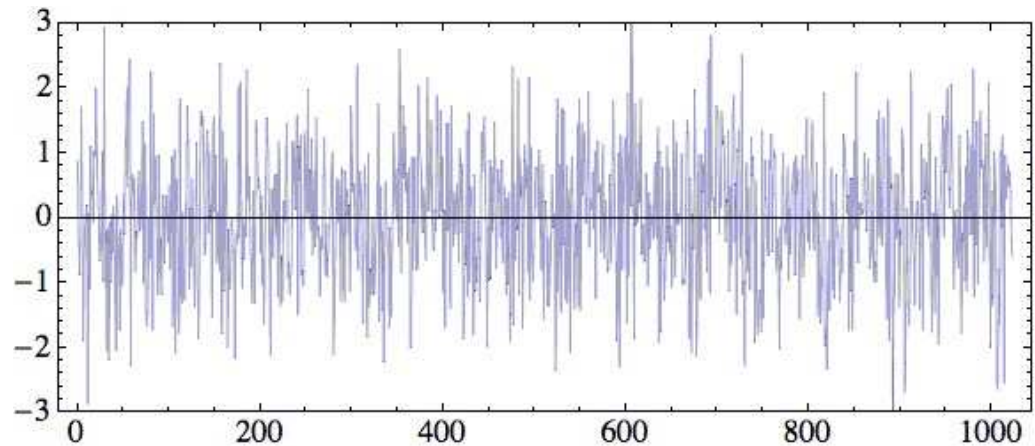
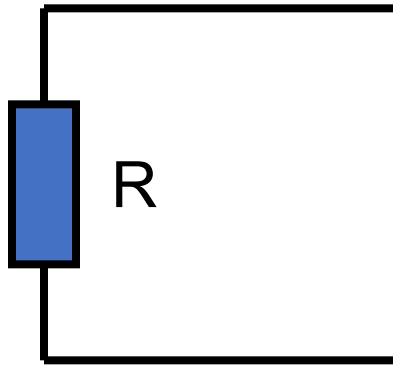


Diverse realizzazioni di un processo di rumore bianco Gaussiano.

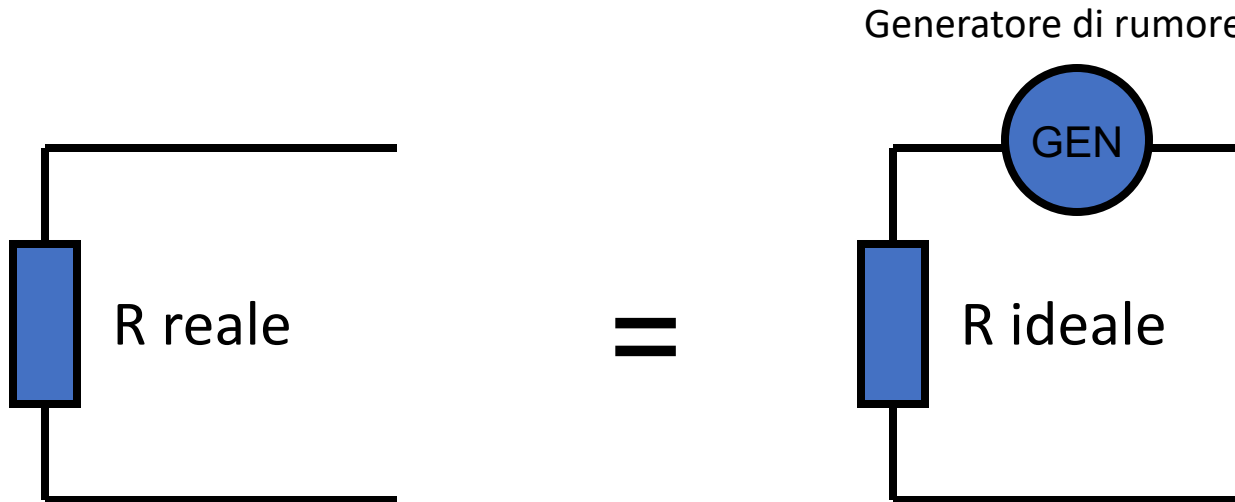
Il rumore è identificato soltanto dalle sue statistiche.



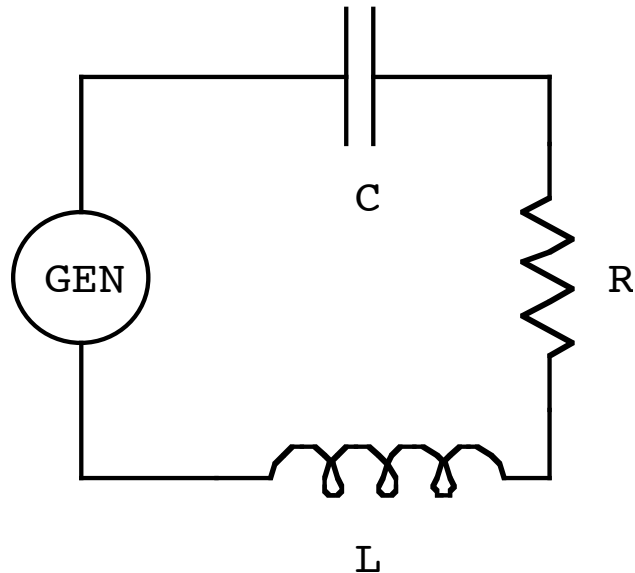
Rumore termico (rumore Johnson)



Dimostrazione di Pierce della formula per il rumore Johnson



Analisi in frequenza fatta aggiungendo un circuito risonante in serie



$$\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\Gamma = \frac{R}{L}$$

$$Q = \frac{\omega_0}{\Gamma} = \frac{\sqrt{L/C}}{R}$$

$$Z(f) = R + i \left(2\pi fL - \frac{1}{2\pi fC} \right)$$

$$|Z(f)|^2 = R^2 + 4\pi^2 f^2 L^2 \left(1 - \frac{f_0^2}{f^2} \right)^2$$

Energia media nell'induttanza

$$\begin{aligned} \left\langle \frac{1}{2} L I^2 \right\rangle &= \frac{L}{2} \int_0^{+\infty} \langle |I(f)|^2 \rangle df = \frac{L}{2} \int_0^{+\infty} \frac{\langle |V(f)|^2 \rangle}{|Z(f)|^2} df \\ &= \frac{L}{2} \int_0^{+\infty} \frac{\langle |V(f)|^2 \rangle}{R^2 + 4\pi^2 f^2 L^2 \left(1 - \frac{f_0^2}{f^2} \right)^2} df \end{aligned}$$

... allora prendendo un Q-valore molto alto ...

$$\left\langle \frac{1}{2} LI^2 \right\rangle = \frac{L}{2} \int_0^{+\infty} \frac{\langle |V(f)|^2 \rangle}{R^2 + 4\pi^2 f^2 L^2 \left(1 - \frac{f_0^2}{f^2} \right)^2} df$$
$$\approx \frac{L}{2} \langle |V(f_0)|^2 \rangle \int_{-\infty}^{+\infty} \frac{df}{R^2 + 16\pi^2 L^2 (f - f_0)^2}$$

$$\frac{(f^2 - f_0^2)^2}{f^2} = \frac{(f - f_0)^2 (f + f_0)^2}{f^2} = (\Delta f)^2 \frac{(2f_0 + \Delta f)^2}{(f_0 + \Delta f)^2} \approx 4(\Delta f)^2$$

$$\int \frac{dx}{a^2 + b^2 x^2} = \frac{1}{ab} \arctan \frac{bx}{a}$$

quindi

$$\int_{-\infty}^{+\infty} \frac{d\Delta f}{R^2 + 16\pi^2 L^2 (\Delta f)^2} = \frac{1}{4\pi LR} \arctan \frac{4\pi L \Delta f}{R} \Big|_{-\infty}^{+\infty} = \frac{1}{4LR}$$

$$\frac{1}{2}kT = \left\langle \frac{1}{2}LI^2 \right\rangle \approx \frac{L}{2} \left\langle |V(f_0)|^2 \right\rangle \frac{1}{4LR}$$

$$\left\langle |V(f_0)|^2 \right\rangle = 4kTR$$

teorema di Nyquist

Si noti che non c'è dipendenza dalla frequenza, si tratta di un rumore bianco.

THERMAL AGITATION OF ELECTRICITY IN CONDUCTORS

BY J. B. JOHNSON

ABSTRACT

*Statistical fluctuation of electric charge exists in all conductors, producing random variation of potential between the ends of the conductor. The effect of these fluctuations has been measured by a vacuum tube amplifier and thermocouple, and can be expressed by the formula $\bar{I}^2 = (2kT/\pi) \int_0^\infty R(\omega) |Y(\omega)|^2 d\omega$. I is the observed current in the thermocouple, k is Boltzmann's gas constant, T is the absolute temperature of the conductor, $R(\omega)$ is the *real* component of impedance of the conductor, $Y(\omega)$ is the transfer impedance of the amplifier, and $\omega/2\pi = f$ represents frequency. The value of Boltzmann's constant obtained from the measurements lie near the accepted value of this constant. The technical aspects of the disturbance are discussed. In an amplifier having a range of 5000 cycles and the input resistance R the power equivalent of the effect is $\bar{V}^2/R = 0.8 \times 10^{-16}$ watt, with corresponding power for other ranges of frequency. The least contribution of *tube noise* is equivalent to that of a resistance $R_e = 1.5 \times 10^3 i_p / \mu$, where i_p is the space current in milliamperes and μ is the effective amplification of the tube.*

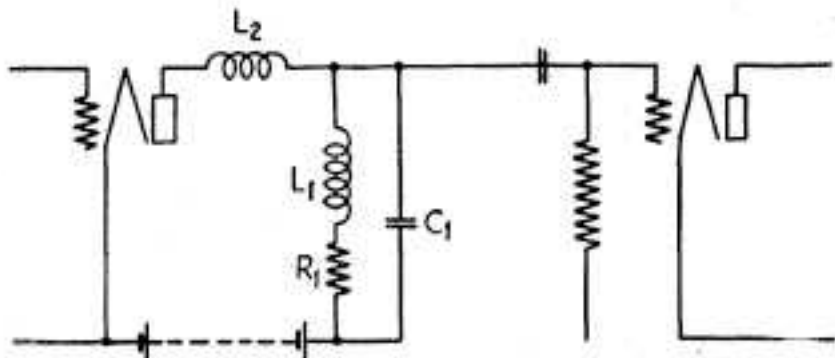


Fig. 2. Diagram of the resonant coupling.

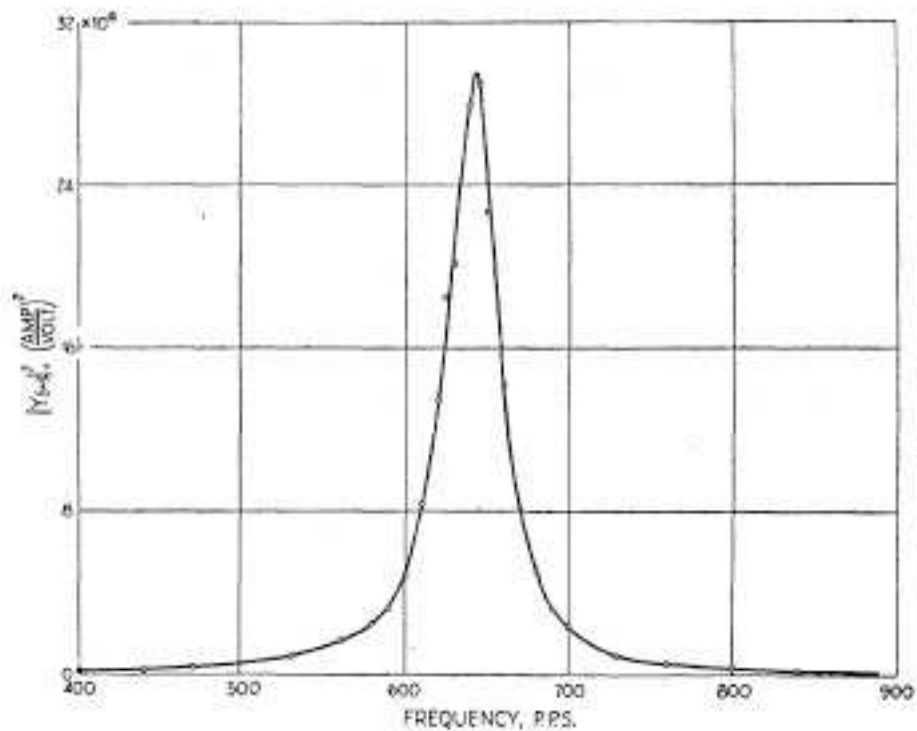


Fig. 3. Typical resonance curve.

Determination of Boltzmann's constant

TABLE I. *Determination of Boltzmann's Constant.*

No.	f p.p.s.	T °K	$R(\omega) \int_0^\infty Y(\omega) ^2 d\omega$ $\times 10^{-8}$	$\int_0^\infty Y(\omega) ^2 R(\omega) d\omega$ $\times 10^{-10}$	$\int_0^\infty Y(\omega) ^2 R(\omega) d\omega$ $\times 10^{-16}$	ΔS %	$\overline{I^2}$ $\times 10^6$	k $\times 10^{10}$
1	1010	298	.526	.213		12.0	2.7	1.27
2	2023	"	.470	.272		26.2	2.8	1.15
3	1418	"	.508	.361		15.6	3.8	1.09
4	"	"	"	.188		42.3	1.8	.99
5	295	"	.548	.252		3.3	3.1	1.18
6	"	"	"	.202		15.8	2.0	.95
7	302	"	"	.221		12.4	2.7	1.18
8	"	"	"	.195		15.3	2.3	1.13
9	653	"	.541	.747		6.8	10.4	1.14
10	"	"	"	.645		12.9	8.6	1.30
11	1418	"	.508	.286		18.5	3.5	1.26
12	"	"	"	.161		41.3	1.7	1.09
13	1465	"	.505	1.93		18.7	21.2	1.14
14	"	"	"	1.75		20.3	19.1	1.14
15	635	"	.541	.594		8.8	8.9	1.46
16	"	"	"	.139		35.0	2.1	1.47
17	"	"	"	.597		10.9	7.8	1.28
18*	643	295	.44 ±		.439	0	11.0	1.38
19	645	297	"		.396	0	11.1	1.49
20	1830	301	"		.913	0	19.8	1.13
21	500-							
	1000	299	"		.831	0	25.9	1.64
22	"	300	"		.662	0	21.5	1.70
23	"	300	"		.832	0	26.0	1.63

* The resonance curve for this determination is that reproduced in Fig. 3.

Harry Nyquist (7/2/1889 – 4/4/1978)

Scienziato che ha lavorato ai laboratori Bell, ha dato contributi fondamentali alla teoria dei sistemi e alla teoria delle telecomunicazioni.

Tra i suoi contributi, ricordiamo:

- è stato autore di 138 brevetti
- miglioramenti a telegrafia e fax
- ha contribuito alla tecnologia della TV analogica
- autore del "criterio di Nyquist" sulla stabilità dei circuiti con retroazione
- il suo nome è legato al teorema del campionamento (teorema di "Nyquist-Shannon")
- ha prodotto la prima spiegazione teorica del rumore Johnson



A hard working problem solver

His daughter Phoebe: “I think one of the ways he succeeded in accomplishing so much is that he was very disciplined. The alarm always rang at 6:45 at our house. He always got right up. While mother fixed his breakfast he got dressed and was out the door punctually at 7:30. You could set your watch by it, but he never hurried. He walked a mile to the train station and rode in to NYC. His return home was just as regular, so we could plan on eating at 6:15 PM. Some days he took the ferry across the Hudson River instead of the tubes under the water because the air was fresher. He did like being outdoors. I have a picture of him of him stretched out on the lawn taking a nap. He always reserved Saturday for household chores and Sunday was church, a good dinner and time to read or think. He frequently had a legal size yellow notebook on his lap and started figuring those equations. I suppose he couldn't let go of a problem until he got it solved.”

(from K. J. Åström)

Keys to Success

- **Talent and creativity**
- **Ability to find good stimulating environments**
- **Curiosity**
- **Energy**
- **Searched fundamental problems**
- **Catch the essence abstract away details**
- **Sound use of mathematics**

(from K. J. Åström)

A Modest Person

- **My impressions and feelings as a stranger in the new country: I don't think it was any different from what it would have been if I had gone to Karlstad or Stockholm.**
- **Distinctions I have received: I have received honors for technical work.**
- **Literary works, books, etc: I have published a few technical papers.**
- **Inventions discoveries, other notable achievements: I have been granted a number of patents.**

(from K. J. Åström)

La dimostrazione di Nyquist

THERMAL AGITATION OF ELECTRIC CHARGE IN CONDUCTORS*

BY H. NYQUIST

ABSTRACT

The electromotive force due to thermal agitation in conductors is calculated by means of principles in thermodynamics and statistical mechanics. The results obtained agree with results obtained experimentally.

DR. J. B. JOHNSON¹ has reported the discovery and measurement of an electromotive force in conductors which is related in a simple manner to the temperature of the conductor and which is attributed by him to the thermal agitation of the carriers of electricity in the conductors. The work to be reported in the present paper was undertaken after Johnson's results were available to the writer and consists of a theoretical deduction of the electromotive force in question from thermodynamics and statistical mechanics.²

Consider two conductors each of resistance R and of the same uniform

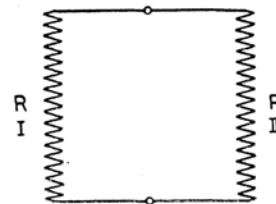


Fig. 1.

temperature T connected in the manner indicated in Fig. 1. The electromotive force due to thermal agitation in conductor I causes a current to be set up in the circuit whose value is obtained by dividing the electromotive force by $2R$. This current causes a heating or absorption of power in conductor II, the absorbed power being equal to the product of R and the square of the current. In other words power is transferred from conductor I to conductor II. In

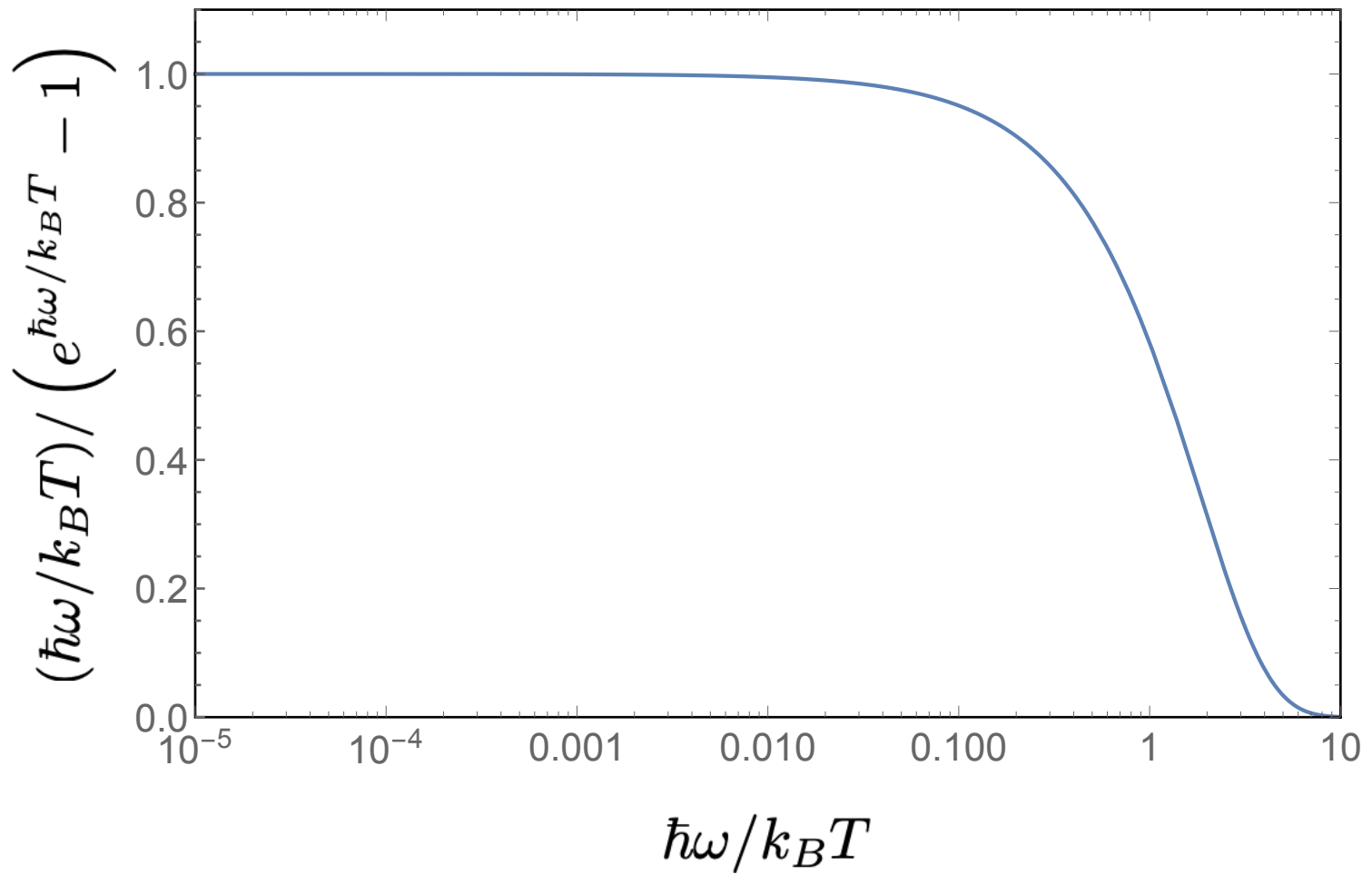
precisely the same manner it can be deduced that power is transferred from conductor II to conductor I. Now since the two conductors are at the same temperature it follows directly from the second law of thermodynamics that the power flowing in one direction is exactly equal to that flowing in the other direction. It will be noted that no assumption has been made as to the nature of the two conductors. One may be made of silver and the other of lead, or one may be metallic and the other electrolytic, etc.

Estensione al caso quantistico

$$k_B T \rightarrow \frac{\hbar\omega}{(e^{\hbar\omega/k_B T} - 1)}$$



$$\langle |V(\omega)|^2 \rangle = 4R \frac{\hbar\omega}{(e^{\hbar\omega/k_B T} - 1)} \xrightarrow{\hbar\omega/k_B T \gg 1} 4R \hbar\omega e^{-\hbar\omega/k_B T}$$



La formula di Nyquist è una forma di "teorema di fluttuazione-dissipazione"

Un altro semplice teorema di fluttuazione-dissipazione si può dimostrare facilmente per un oscillatore classico

Uguagliando l'energia termica media e l'energia potenziale media, si trova

$$\frac{1}{2}k_B T = \frac{1}{2}m\omega_0^2 \langle x^2 \rangle$$



$$\langle x^2 \rangle = \frac{k_B T}{m\omega_0^2}$$

Esempio

Rumore Johnson totale di una resistenza $R = 1 \text{ M}\Omega$ @ 300 K
con 1 KHz di larghezza di banda

$$\sigma_V^2 = \langle |V|^2 \rangle = 4kTRB$$

$$\begin{aligned}\sigma_V^2 &= \langle |V(f)|^2 \rangle \cdot (1 \text{ KHz}) \\ &= 4 \cdot (1.38 \cdot 10^{-23} \text{ J} \cdot \text{K}^{-1}) \cdot (300 \text{ K}) \cdot (1 \text{ M}\Omega) \cdot (1 \text{ KHz}) \\ &\approx 1.7 \cdot 10^{-11} \text{ V}^2\end{aligned}$$

$$\sigma_V \approx 4 \cdot 10^{-6} \text{ V} = 4 \mu\text{V}$$

PROJECTS/PROGRAMS

Johnson Noise Thermometry



Summary

Johnson noise thermometry (JNT) is a primary temperature measurement technique based on the fundamental properties of thermal fluctuations in conductors. We measure these fluctuations with respect to quantum electronic noise sources and/or other thermal noise sources. The results are useful for both temperature scale metrology and for the development of reliable thermometers for harsh environments. The NIST-developed spectral JNT methods link the measured thermal voltage directly to the as-maintained electrical units via the AC Josephson Voltage Standard. This new methodology conveys key advantages from a metrology standpoint and will extend the practical application range for JNT systems.

ORGANIZATIONS

Physical Measurement Laboratory
Sensor Science Division
Thermodynamic Metrology Group

NIST STAFF

Weston L. Tew

DESCRIPTION

The NIST Gaithersburg JNT project produces data useful for verification of the International Temperature Scale of 1990 (ITS-90) over a temperature range from 500 K to 930 K. This range overlaps those covered by other thermodynamic methods, most notably the upper-most range of Acoustic Thermometry and the lower-most range of absolute Radiation Thermometry.

The JNT results provide independent data for scale validation in these overlapping ranges. We expect future impacts in advancing the application of JNT temperature measurements in new demanding environments, such as high-temperature nuclear reactors.

Objective:


To obtain data on JNT temperatures with respect to the ITS-90 in these important overlapping ranges with uncertainties comparable to the best Radiation Thermometry methods.




Johnson Noise Thermometer

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METROSOL
Metrology Solutions
Thursday December 9, 2021



Check out the latest news about the development of our JNT project on Facebook



Commercial Johnson noise thermometers

The World's first practical Johnson Noise Thermometer
developed byMETROSOL Limited

Johnson noise thermometry (JNT) is a primary temperature measurement technique based on the fundamental properties of thermal fluctuations in conductors.

A Johnson noise thermometer never needs calibrating and is insensitive to the condition of the sensor material, so is ideally suited to long-term temperature measurements in harsh environments (such as nuclear reactor coolant circuits, nuclear waste management and storage) or where low drift is required (such as high temperature reference standards in metrology or the annealing of single crystal turbine blades in the aerospace industry).



JOHNSON NOISE THERMOMETERS why use one?

At present all thermometers in use (thermocouples, platinum resistance thermometers, thermocouples) are secondary thermometers and are therefore prone to drift. They don't actually measure temperature directly, instead a property of the sensor (such as resistance or EMF) is measured. The property is related to temperature by a calibration process. Either the



4KTRAF

Dimostrazione del teorema di Nyquist per mezzo del teorema di Campbell

- Richiede la conoscenza della distribuzione di Poisson;
- Richiede conoscenze basilari sulla conduzione nei metalli

Semplice schematizzazione semiclassica della conduzione (modello di Drude-Lorenz)

Consideriamo un gas classico di elettroni quasi liberi all'interno del conduttore metallico: dal principio di equipartizione dell'energia sappiamo che la loro energia cinetica media deve essere

$$\text{energia cinetica media} = \frac{1}{2} m \langle v_t^2 \rangle = \frac{3}{2} kT$$

e dunque

$$\langle v_t^2 \rangle^{1/2} = \sqrt{\frac{3kT}{m}}$$

a temperatura ambiente (300 K)

$$\langle v_t^2 \rangle^{1/2} \approx 1.2 \cdot 10^5 \frac{m}{s}$$

Se applichiamo una differenza di potenziale V ai capi di un conduttore di lunghezza L , allora c'è un campo elettrico $E=V/L$ (E = modulo del campo elettrico), e se indichiamo con q la carica elettronica, allora su ciascun elettrone agisce una forza (in modulo)

$$F = qE = \frac{qV}{L}$$

a cui corrisponde un'accelerazione

$$a = \frac{qV}{mL}$$

La variazione media di velocità tra due urti si ottiene utilizzando la distribuzione degli intervalli

$$\langle \Delta v \rangle = \int_0^{\infty} \Delta v(t) e^{-t/\tau} \frac{dt}{\tau} = \int_0^{\infty} a t e^{-t/\tau} \frac{dt}{\tau} = a\tau$$

Quindi si trova la velocità di deriva

$$v = a\tau = \frac{qE\tau}{m} = \frac{qV\tau}{mL}$$

e la densità di corrente

$$J = nqv = \frac{nq^2V\tau}{mL}$$

e infine la corrente

$$I = JS = \frac{nq^2V\tau}{mL} S = \left(\frac{nq^2\tau}{m} \right) \left(\frac{S}{L} \right) V = \frac{V}{R}$$

$$I = \frac{nq^2\tau}{m} \left(\frac{S}{L} \right) V = \sigma \left(\frac{S}{L} \right) V$$



$$\tau \approx \frac{\ell}{\langle v_t^2 \rangle^{1/2}};$$

$$\sigma = \frac{nq^2\tau}{m} \approx \frac{nq^2\ell}{m \langle v_t^2 \rangle^{1/2}}$$

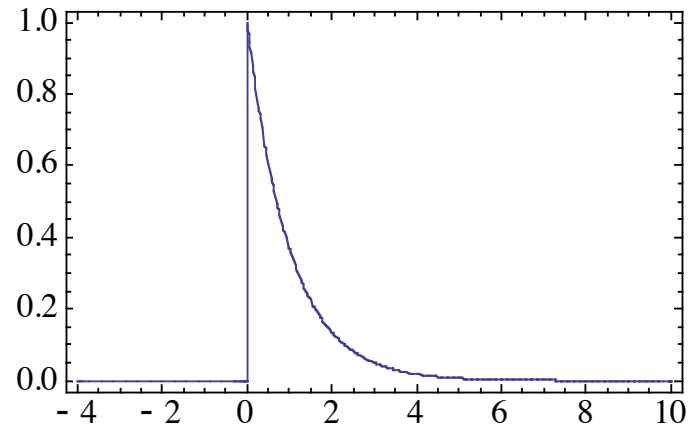
$$\rho = \frac{m}{nq^2\tau} \approx \frac{m \langle v_t^2 \rangle^{1/2}}{nq^2\ell}$$

Il teorema di Campbell

Consideriamo ora un sistema con una risposta impulsiva $h(t)$

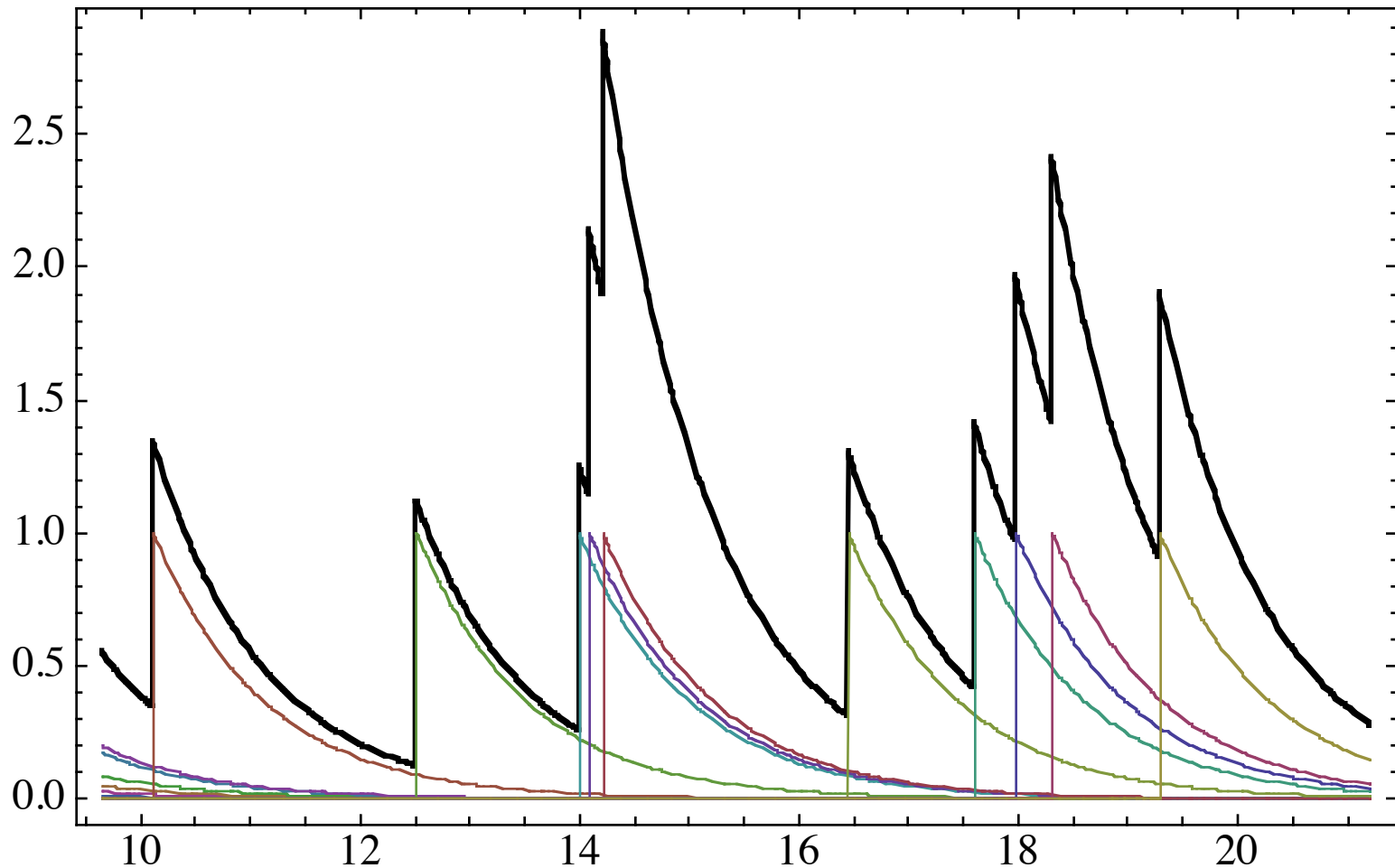
Esempio: nel caso del circuito RC abbiamo visto che

$$h(t) = \begin{cases} 0 & \text{se } t < 0 \\ \frac{e^{-t/RC}}{RC} & \text{se } t \geq 0 \end{cases}$$

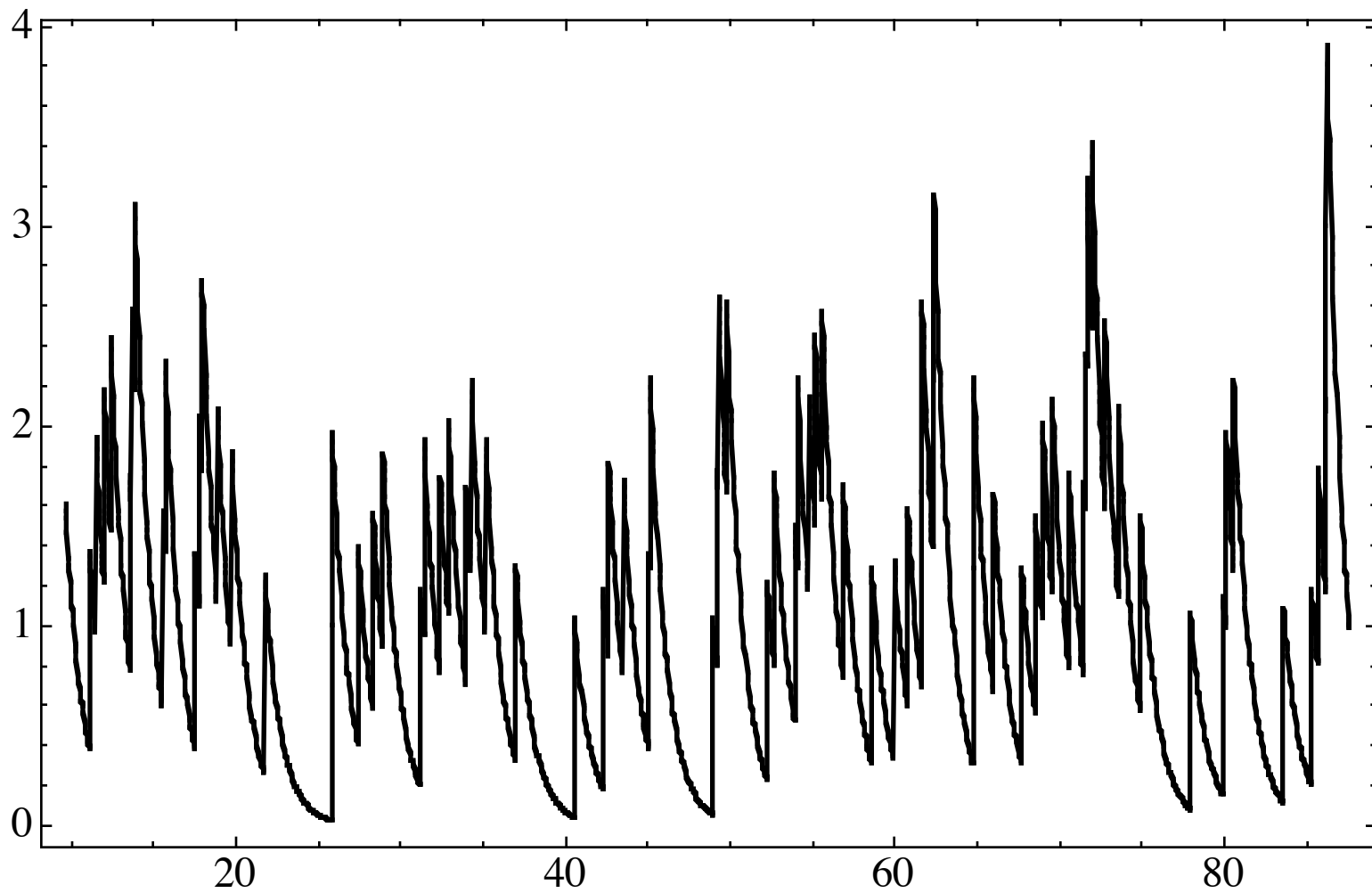


$$V_C(t) = \int_{-\infty}^{+\infty} V(s)h(t-s)ds$$

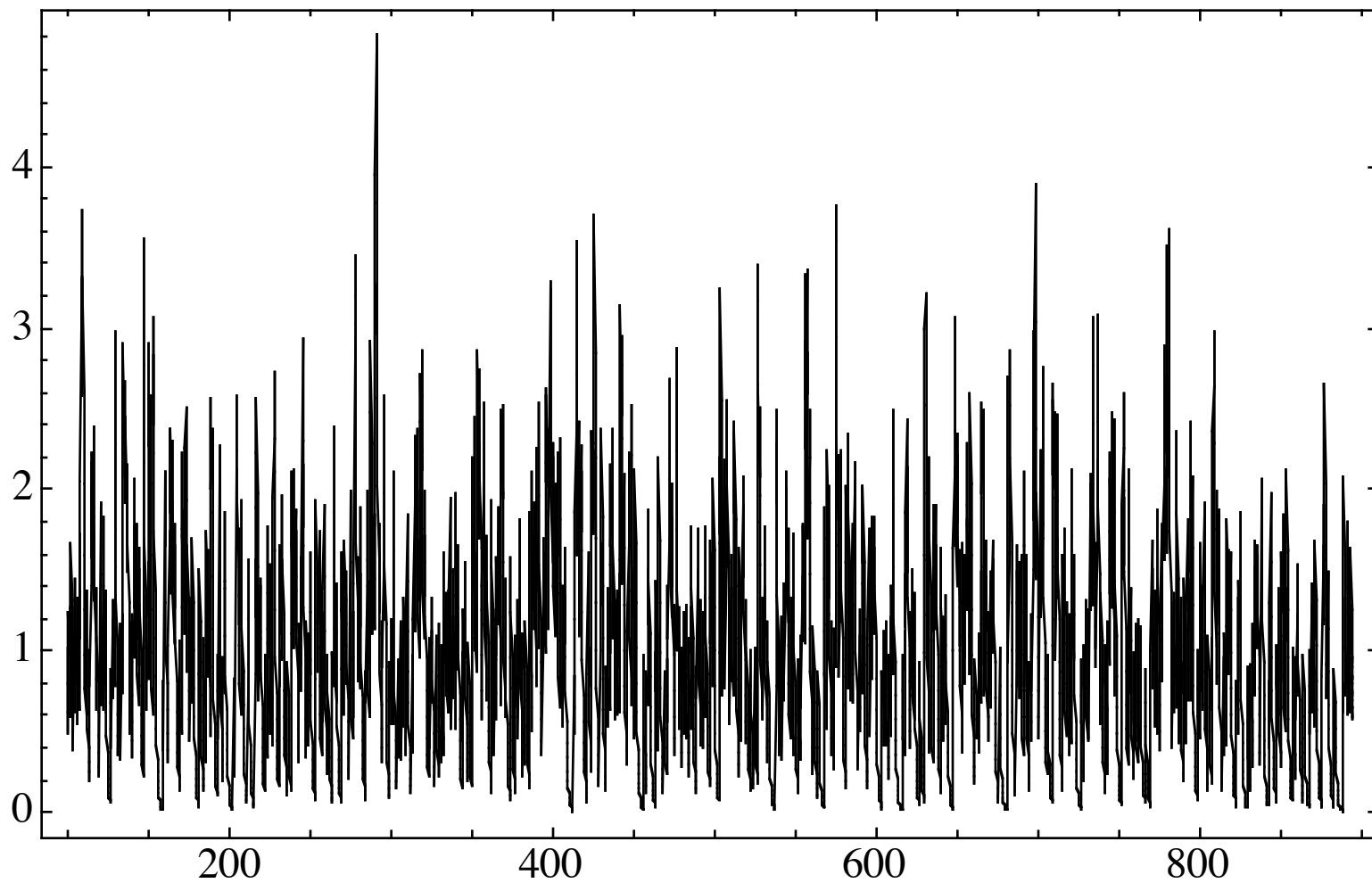
$$V_C(t) = \int_{-\infty}^{+\infty} V(s)h(t-s)ds \quad \text{limite di} \quad V_C(t) \approx \sum_k V(t_k)h(t-t_k)$$



Impulsi distribuiti Poissonianamente - I



Impulsi distribuiti Poissonianamente - II



Valore medio e varianza della risposta del sistema ad treno di impulsi distribuiti Poissonianamente

a numero di impulsi per unità di tempo

S ampiezza di ciascun impulso

$adt'Sh(t'' - t')$ risposta media al tempo t'' dovuta ad impulsi che arrivano al tempo t'

Risposta media (es. tensione media ai capi del condensatore nel circuito RC) al tempo t'' :

$$\langle R \rangle = \int_{-\infty}^{t''} adt'Sh(t'' - t') = aS \int_{-\infty}^{t''} h(t'' - t')dt'$$

ponendo $t = t'' - t'$

$$\langle R \rangle = aS \int_0^{\infty} h(t) dt$$

Analogamente, notando che la varianza del numero di impulsi che arrivano nel tempo dt è adt'

$$\begin{aligned} \text{var } R &= \int_{-\infty}^{t''} a dt' S^2 h^2(t'' - t') = aS^2 \int_{-\infty}^{t''} h^2(t'' - t') dt' \\ &= aS^2 \int_0^{\infty} h^2(t) dt \end{aligned}$$

Le formule

$$\langle R \rangle = aS \int_0^{\infty} h(t) dt$$

$$\text{var } R = aS^2 \int_0^{\infty} h^2(t) dt$$

costituiscono insieme il *teorema di Campbell*. Si dimostra inoltre facilmente che vale anche l'estensione

$$\langle R \rangle = a \langle S \rangle \int_0^{\infty} h(t) dt$$

$$\text{var } R = a \langle S^2 \rangle \int_0^{\infty} h^2(t) dt$$

Si noti ora che ...

$$\int_0^{\infty} h^2(t) dt = \int_{-\infty}^{+\infty} h^2(t) dt = \int_{-\infty}^{+\infty} |H(f)|^2 df = 2 \int_0^{\infty} |H(f)|^2 df$$

$h(t) = 0$ se $t < 0$

teor. di Parseval

$H(f)$ è pari perché $h(t)$
è reale

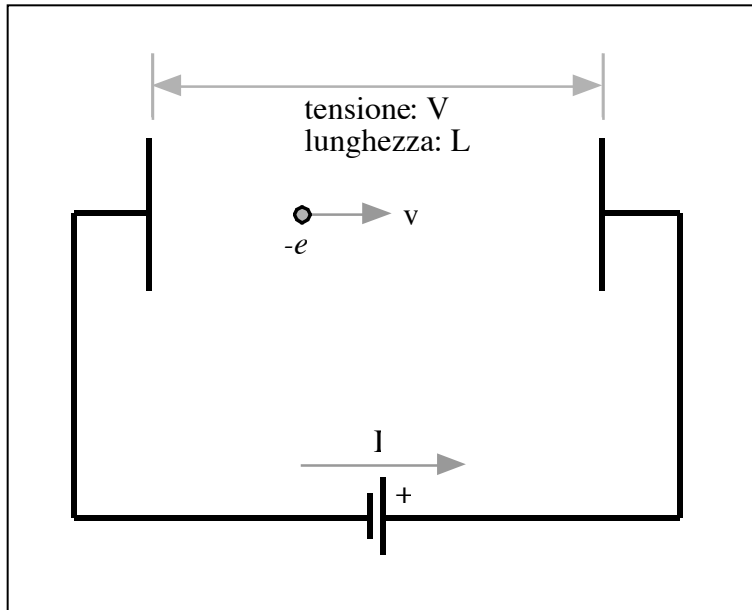
quindi

$$\begin{aligned}\text{var } R &= \int_0^{\infty} \langle |R(f)|^2 \rangle df \\ &= a \langle S^2 \rangle \int_0^{\infty} h^2(t) dt = 2a \langle S^2 \rangle \int_0^{\infty} |H(f)|^2 df\end{aligned}$$

Notando che la risposta impulsiva approssima una funzione delta (nel limite $RC \rightarrow 0$), allora si ottiene $H(f) = 1$, e quindi la varianza della risposta per unità di frequenza del sistema è

$$\langle |R(f)|^2 \rangle = 2a \langle S^2 \rangle$$

Ultimo preliminare: una versione semplificata del teorema di Shockley-Ramo



- Forza che agisce sull'elettrone

$$qV/L$$

- Lavoro fatto per uno spostamento Δx

$$\frac{qV}{L} \Delta x$$

- Lavoro fatto dal generatore di tensione

$$I \cdot V \Delta t$$

$$\frac{qV}{L} \Delta x = I \cdot V \Delta t$$

poiché $v = \frac{\Delta x}{\Delta t}$ allora

$$I = q \frac{v}{L}$$

Applicazione del teorema di Campbell al rumore Johnson

Consideriamo gli impulsi di carica associati agli spostamenti degli elettroni tra una collisione e l'altra all'interno di una resistenza R che ha lunghezza L e sezione A

a = frequenza delle collisioni

S = ampiezza di un singolo impulso di carica

In questo modo il prodotto aS è una corrente, e dal teor. di Campbell si trova

$$\langle |R(f)|^2 \rangle = 2aS^2 \quad \longrightarrow \quad \langle |\Delta I(f)|^2 \rangle = 2a \langle S^2 \rangle$$

Stima del numero di impulsi per unità di tempo a

n = densità media degli elettroni nel conduttore

τ = tempo medio tra due collisioni (per un singolo el.)

Allora numero di el. nella resistenza = nAL

Numero di collisioni per unità di tempo $a = nAL/\tau$

Stima dell'ampiezza quadratica media

$$\langle S^2 \rangle$$

Corrente dovuta al movimento di un elettrone

$$I = q \frac{v}{L}$$

Singolo impulso di carica per un movimento che dura Δt da una collisione all'altra


$$S = q \frac{v}{L} \Delta t$$

Allora

v e Δt sono statisticamente indipendenti

$$\langle S^2 \rangle = \left\langle \left(\frac{qv\Delta t}{L} \right)^2 \right\rangle = \frac{q^2}{L^2} \langle v^2 \rangle \langle (\Delta t)^2 \rangle$$

Però noi sappiamo che $\frac{1}{2}m\langle v^2 \rangle = \frac{kT}{2}$

 $\langle v^2 \rangle = \frac{kT}{m}$

$$\langle (\Delta t)^2 \rangle = \int_0^{\infty} (\Delta t)^2 \frac{1}{\tau} e^{-\Delta t/\tau} d\Delta t = \tau^2 \int_0^{\infty} x^2 e^{-x} dx = 2\tau^2$$

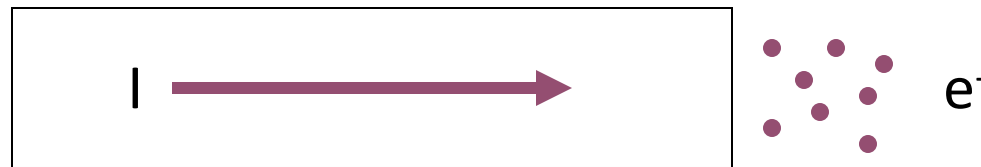
quindi

$$\langle S^2 \rangle = \frac{q^2}{L^2} \langle v^2 \rangle \langle (\Delta t)^2 \rangle = \frac{q^2}{L^2} \cdot \frac{kT}{m} \cdot 2\tau^2 = 2 \frac{q^2 \tau^2 kT}{mL^2}$$

Infine

$$\begin{aligned}\langle |I(f)|^2 \rangle &= 2a \langle S^2 \rangle = 2 \frac{nAL}{\tau} \cdot 2 \frac{q^2 \tau^2 kT}{mL^2} \\ &= 4 \frac{nq^2 \tau}{m} \cdot \frac{A}{L} kT = 4\sigma \cdot \frac{A}{L} kT \\ &= 4 \frac{kT}{R} \\ \langle |V(f)|^2 \rangle &= R^2 \langle |I(f)|^2 \rangle = 4kTR\end{aligned}$$

Applicazione del teorema di Campbell allo shot noise



carattere “granulare” della corrente elettrica

$$a = \frac{I}{q}$$

$$S^2 = q^2$$

quindi $\langle |I(f)|^2 \rangle = 2aS^2 = 2\frac{I}{q}q^2 = 2Iq$

Esempio

Shot noise totale per una corrente $I = 0.1$ A con 1 KHz di larghezza di banda

$$\sigma_I^2 = \langle |I|^2 \rangle = 2qIB$$

$$\begin{aligned}\sigma_I^2 &= \langle |I(f)|^2 \rangle \cdot (1 \text{ KHz}) \\ &\approx 2 \cdot (0.1 \text{ A}) \cdot (1.6 \cdot 10^{-19} \text{ C}) \cdot (1 \text{ KHz}) \\ &= 3.2 \cdot 10^{-17} \text{ A}^2\end{aligned}$$

$$\sigma_I \approx 6 \cdot 10^{-9} \text{ A} = 6 \text{ nA}; \quad \frac{\sigma_I}{I} \approx 6 \cdot 10^{-8}$$

Il teorema di Shockley-Ramo

Currents to Conductors Induced by a Moving Point Charge

W. SHOCKLEY

Bell Telephone Laboratories, Inc., New York, N. Y.

(Received May 14, 1938)

General expressions are derived for the currents which flow in the external circuit connecting a system of conductors when a point charge is moving among the conductors. The results are applied to obtain explicit expressions for several cases of practical interest.

636

VOLUME 9, OCTOBER, 1938

JOURNAL OF APPLIED PHYSICS

Currents Induced by Electron Motion*

SIMON RAMO†, ASSOCIATE MEMBER, I.R.E.

Summary—A method is given for computing the instantaneous current induced in neighboring conductors by a given specified motion of electrons. The method is based on the repeated use of a simple equation giving the current due to a single electron's movement and is believed to be simpler than methods previously described.

584

Proceedings of the I.R.E.

September, 1939

La formula di reciprocità di Green

Supponiamo che le cariche q_1, q_2, \dots in posizioni indicate con 1, 2, ... producano i potenziali $V_1, V_2 \dots$ nelle medesime posizioni. Analogamente, le cariche q'_1, q'_2, \dots producono i potenziali $V'_1, V'_2 \dots$. Allora

$$V_n = \sum_{k \neq n} \frac{q_k}{r_{kn}}$$

e quindi troviamo

$$\sum_n q'_n V_n = \sum_{k, n (k \neq n)} \frac{q'_n q_k}{r_{kn}} = \sum_n q_n V'_n$$

Se ora consideriamo cariche estese Q_k e le decomponiamo nelle cariche puntuali componenti, si vede subito che si può usare la formula

$$\sum_n q'_n V_n = \sum_n q_n V'_n$$

per ottenere

$$\sum_k Q'_k V_k = \sum_k Q_k V'_k$$

Ora consideriamo i casi speciali

$$Q_1 = 1; \quad Q_2 = Q_3 = Q_4 = \dots = 0 \quad \Rightarrow \quad V_1 = p_{11}; \quad V_2 = p_{12}; \quad \dots$$

$$Q'_1 = 0; \quad Q'_2 = 1; \quad Q'_3 = Q'_4 = \dots = 0 \quad \Rightarrow \quad V'_1 = p_{21}; \quad V'_2 = p_{22}; \quad \dots$$

da cui si trova

$$\sum_k Q'_k V_k = \sum_k Q_k V'_k \quad \Rightarrow \quad \sum_k Q'_k V_k = p_{12}; \quad \sum_k Q_k V'_k = p_{21}$$

e quindi

$$p_{12} = p_{21}$$

vale a dire che il potenziale indotto sull'elettrodo 2 da una carica unitaria sull'elettrodo 1 è uguale al potenziale indotto sull'elettrodo 1 da una carica unitaria sull'elettrodo 2 (con tutti gli altri elettrodi scarichi).

Utilizzando il principio di sovrapposizione troviamo allora:

$$V_1 = p_{11}Q_1 + p_{12}Q_2 + p_{13}Q_3 + \dots$$

$$V_2 = p_{21}Q_1 + p_{22}Q_2 + p_{23}Q_3 + \dots$$

...

e in seguito, invertendo il sistema lineare troviamo

$$Q_1 = C_{11}V_1 + C_{12}V_2 + C_{13}V_3 + \dots$$

$$Q_2 = C_{21}V_1 + C_{22}V_2 + C_{23}V_3 + \dots$$

...

dove i coefficienti C_{ij} sono le capacità interelettrodiche, e vale la relazione

$$C_{ij} = C_{ji}$$

dal momento che l'inversa di una matrice simmetrica è ancora simmetrica.

Il teorema di Shockley-Ramo

Ora consideriamo un'altra coppia di situazioni in cui si considera una carica elementare in posizione \mathbf{r} e uno specifico elettrodo (2)

Situazione 1:

$$Q_1 = 0; \quad V_1 = V_1(\mathbf{r}); \quad V_2 = V; \quad V_3 = V_4 = \dots = 0$$

Situazione 2:

$$Q'_1 = q(\mathbf{r}) = q; \quad Q'_2 = Q; \quad V'_2 = V'_3 = V'_4 = \dots = 0$$

Allora

$$\sum_k Q'_k V_k = \sum_k Q_k V'_k \quad \Rightarrow \quad qV_1(\mathbf{r}) = QV$$

rappresentazione dei conduttori nell'articolo di Shockley

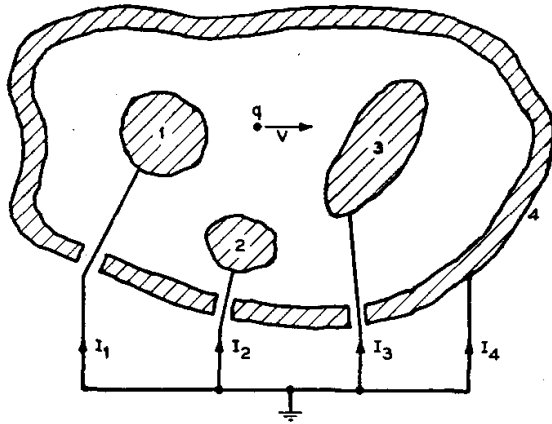


FIG. 1. Schematic representation of conductors and currents.

$$qV_1(\mathbf{r}) = QV$$



derivata
rispetto al
tempo

rappresentazione dei conduttori nell'articolo di Ramo

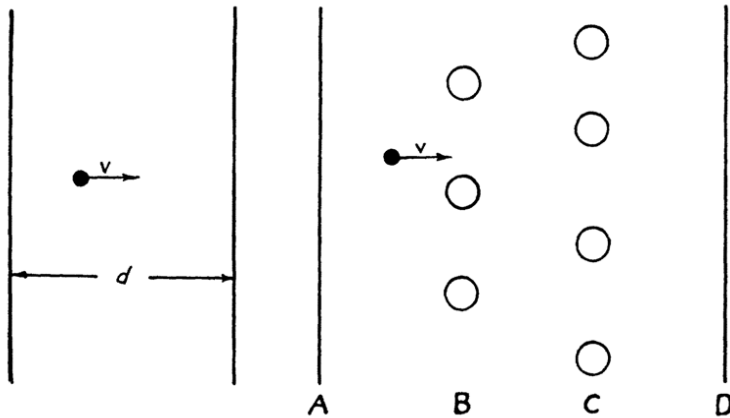


Fig. 1

Fig. 2

$$\begin{aligned} q \frac{d}{dt} V_1(\mathbf{r}) &= q \nabla V_1 \cdot \frac{d\mathbf{r}}{dt} \\ &= q \nabla V_1 \cdot \mathbf{v} \\ &= V \frac{dQ}{dt} = VI \end{aligned}$$

Riassumendo:

$$q\mathbf{E} \cdot \mathbf{v} = VI$$

e quindi, se trattiamo un caso quasi 1-dimensionale, campo e velocità sono paralleli, e inoltre

$$E = V/L$$

e perciò ritroviamo

$$I = \frac{qv}{L}$$

Il teorema di Shockley-Ramo ha molte applicazioni nello studio del trasporto di carica.

Nota sull'applicazione del teorema di Shockley-Ramo alle coppie elettrone-lacuna generate nei fotodiodi:

nel caso di una coppia elettrone-lacuna, ciascuna delle cariche contribuisce al percorso che corrisponde ad una frazione della lunghezza totale.

La somma dei due contributi è equivalente allo spostamento di un'unica carica attraverso l'intera lunghezza L .

In un caso del genere la coppia contribuisce alla carica totale trasportata come se fosse un solo portatore di carica.

Funzione di autocorrelazione del rumore bianco

$$S(\omega) = \sigma_0^2 = \text{costante}$$



teorema di Wiener-Kintchine

$$\begin{aligned} R(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} f(t) f(t + \tau) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S(\omega) e^{i\omega\tau} d\omega \\ &= \frac{\sigma_0^2}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega\tau} d\omega = \sigma_0^2 \delta(\tau) \end{aligned}$$

La funzione di autocorrelazione del rumore bianco diverge per $\tau=0$.

Esempio di processo stazionario, non-ergodico

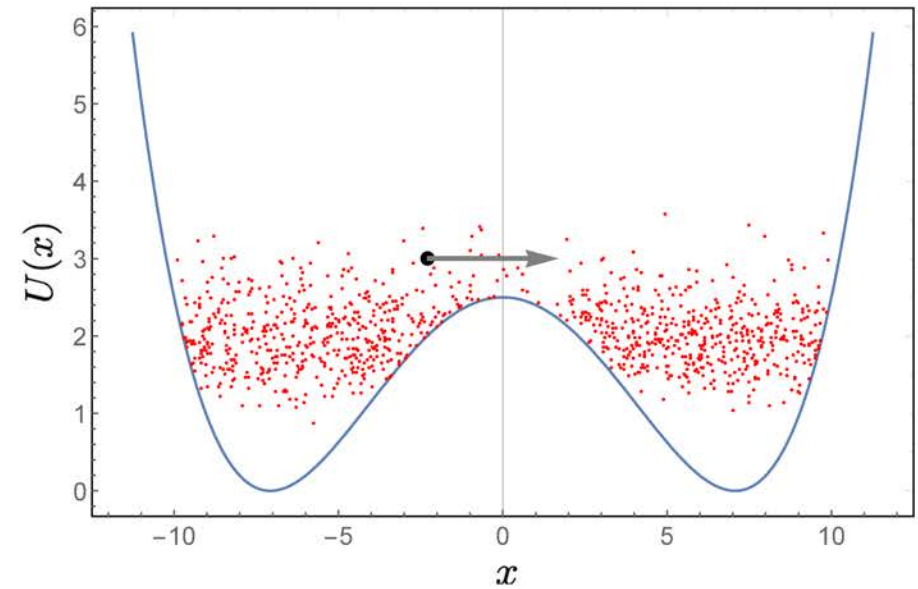
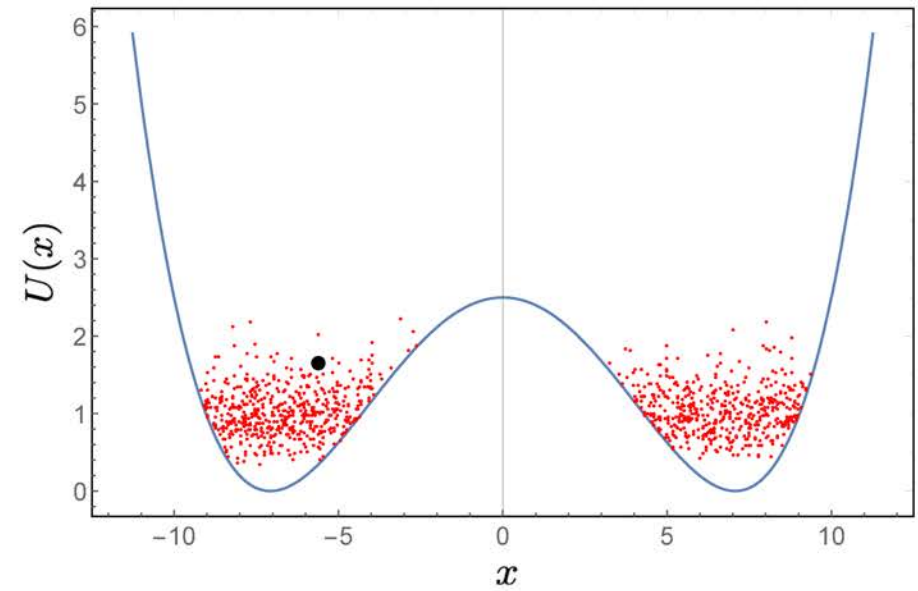


Figure 2.6: Upper panel: potential energy vs. position (arbitrary units) for the example of non-ergodic motion considered in the main text. There are two potential wells, and when the particle is in one of the two wells and the temperature is too low it cannot escape from the well. The red dots outline the position-energy distribution at the temperature of the system and the black dot represents the current position of the particle and its total energy: it has enough kinetic energy to move freely in the well, but not enough to escape. The thermal fluctuations are too small to reach a total energy sufficient to overcome the barrier between the two wells. Lower panel: on the contrary, when the temperature is high enough, the fluctuations are large and the distribution is wider; the particle can jump to the neighboring well and in the long run the particle explores evenly the two potential wells.

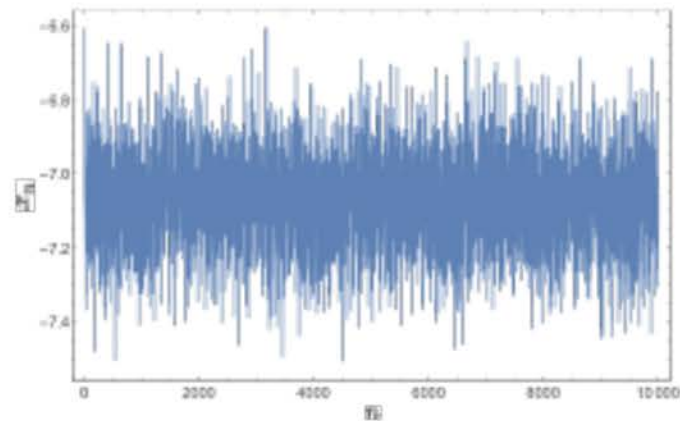


Figure 2.7: Simulation of particle motion in the potential energy landscape shown in figure 2.6, at low temperature. The initial position of the particle is $x(0) = -5$, and after rapidly converging close to the equilibrium position, the noise process fluctuates about a mean position close to $x \approx -7.1$, which is the approximate center of the potential well on the right in figure 2.6.

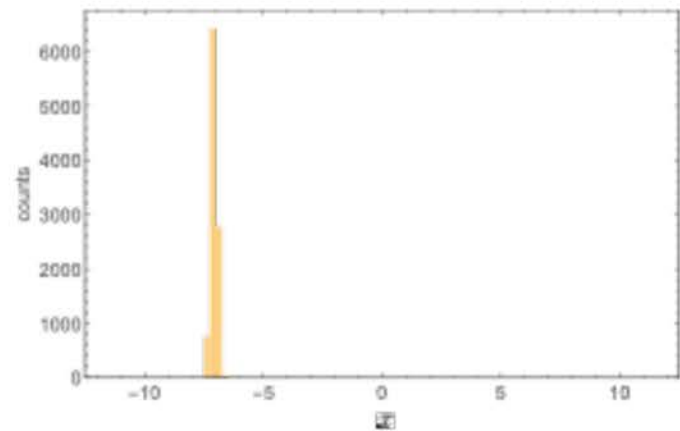


Figure 2.8: Histogram of 10000 samples of the noise process shown in figure 2.7. At low temperature, the particle is bound to its starting potential well.

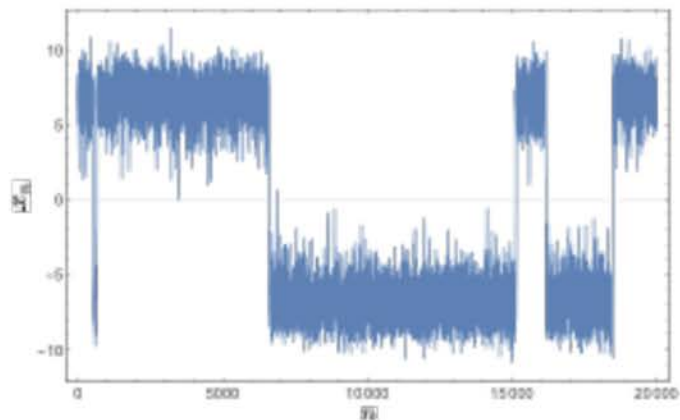


Figure 2.9: Simulation of particle motion in the potential energy landscape shown in figure 2.6, at high temperature. The initial position of the particle is again $x(0) = 5$. From time to time, the particle jumps between the potential wells.

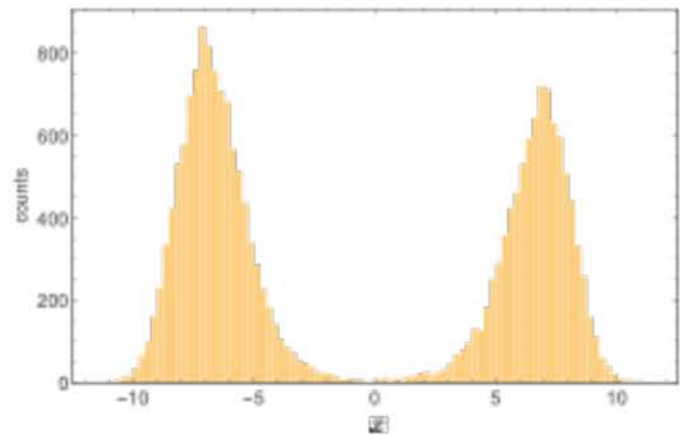


Figure 2.10: Histogram of 10000 samples of the noise process shown in figure 2.9. At high temperature, the particle jumps between the potential wells, and this shows up as a double peak in the histogram.

Moto Browniano

Produced by the Institute of Physics and the National STEM Learning Centre and Network (<https://www.stem.org.uk/>), this video shows the movement of particles by Brownian motion in a suspension containing micrometre diameter polystyrene spheres. (<https://www.youtube.com/watch?v=PzssJDZn9xI>)



**5. *Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen;*
von A. Einstein.**

...

§ 4. Über die ungeordnete Bewegung von in einer Flüssigkeit suspendierten Teilchen und deren Beziehung zur Diffusion.

...

$$(1) \quad \frac{\partial f}{\partial t} = D \frac{\partial^2 f}{\partial x^2}.$$

Dies ist die bekannte Differentialgleichung der Diffusion, und man erkennt, daß D der Diffusionskoeffizient ist.

...

...

§ 5. **Formel für die mittlere Verschiebung suspendierter Teilchen.
Eine neue Methode zur Bestimmung der wahren Größe der Atome.**

...

$$N = \frac{t}{\lambda_x^2} \cdot \frac{R T}{3 \pi k P} .$$

...

Möge es bald einem Forscher gelingen, die hier aufgeworfene, für die Theorie der Wärme wichtige Frage zu entscheiden!

Bern, Mai 1905.

(Eingegangen 11. Mai 1905.)

PHYSIQUE. — *Sur la théorie du mouvement brownien.*

Note de M. **P. LANGEVIN**, présentée par M. Mascart.

I. Le très grand intérêt théorique présenté par les phénomènes de mouvement brownien a été signalé par M. Gouy (1) : on doit à ce physicien d'avoir formulé nettement l'hypothèse qui voit dans ce mouvement continu des particules en suspension dans un fluide un écho de l'agitation thermique moléculaire, et de l'avoir justifiée expérimentalement, au moins de manière qualitative, en montrant la parfaite permanence du mouvement brownien et son indifférence aux actions extérieures lorsque celles-ci ne modifient pas la température du milieu.

Une vérification quantitative de la théorie a été rendue possible par M. Einstein (2), qui a donné récemment une formule permettant de prévoir quel est, au bout d'un temps donné τ , le carré moyen $\overline{\Delta_x^2}$ du déplacement Δ_x d'une particule sphérique dans une direction donnée x par suite du mouvement brownien dans un liquide, en fonction du rayon a de la particule, de la viscosité μ du liquide et de la température absolue T . Cette formule est

$$(1) \quad \overline{\Delta_x^2} = \frac{RT}{N} \frac{1}{3\pi\mu a} \tau,$$

où R est la constante des gaz parfaits relative à une molécule-gramme et N

(1) GOUY, *Journ. de Phys.*, 2^e série, t. VII, 1888, p. 561; *Comptes rendus*, t. CIX, 1889, p. 102.

(2) A. EINSTEIN, *Ann. d. Physik*, 4^e série, t. XVII, 1905, p. 549; *Ann. d. Physik*, 4^e série, t. XIX, 1906, p. 371.

Il modello di Langevin del processo di Ornstein-Uhlenbeck

$$m \frac{dv}{dt} = -\gamma v + \xi v(t) \quad \text{equazione di Langevin}$$

forza d'attrito

processo di rumore bianco
Gaussiano (processo di
Wiener)

L'equazione di Langevin è un esempio di equazione differenziale stocastica

$$m \frac{dv}{dt} = -\gamma v + \xi v(t)$$



media di
ensemble

$$m \frac{d\langle v \rangle}{dt} = -\gamma \langle v \rangle$$



prima integrazione per ottenere
una soluzione parziale (velocità
media)

$$\langle v(t) \rangle_{v_0} = v_0 \exp\left(-\frac{\gamma}{m} t\right)$$

$$m \frac{dv}{dt} = -\gamma v + \xi v(t)$$



media di ensemble dopo avere moltiplicato per x

$$m \left\langle x \frac{dv}{dt} \right\rangle = -\gamma \langle vx \rangle.$$

$$xv = \frac{1}{2} \frac{dx^2}{dt}$$

$$x \frac{dv}{dt} = x \frac{d^2x}{dt^2} = \frac{d}{dt} \left(x \frac{dx}{dt} \right) - \left(\frac{dx}{dt} \right)^2 = \frac{1}{2} \frac{d^2x^2}{dt^2} - v^2$$



per il prossimo passo si utilizzano le due identità differenziali a sinistra

$$\frac{1}{2} \left\langle \frac{d^2x^2}{dt^2} \right\rangle - \langle v^2 \rangle = -\frac{\gamma}{2m} \left\langle \frac{dx^2}{dt} \right\rangle.$$

$$\frac{1}{2} \left\langle \frac{d^2 x^2}{dt^2} \right\rangle - \langle v^2 \rangle = -\frac{\gamma}{2m} \left\langle \frac{dx^2}{dt} \right\rangle.$$

$$\frac{1}{2} m \langle v^2 \rangle = \frac{k_B T}{2}.$$



qui si usa il teorema di equipartizione

$$\frac{d^2 \langle x^2 \rangle}{dt^2} + \frac{\gamma}{m} \frac{d \langle x^2 \rangle}{dt} = \frac{2k_B T}{m}$$



si modifica l'equazione per mezzo di un fattore integrante

$$\frac{d}{dt} \left(e^{(\gamma/m)t} \frac{d \langle x^2 \rangle}{dt} \right) = e^{(\gamma/m)t} \frac{2k_B T}{m}$$

$$\frac{d}{dt} \left(e^{(\gamma/m)t} \frac{d\langle x^2 \rangle}{dt} \right) = e^{(\gamma/m)t} \frac{2k_B T}{m}$$



si integra con la condizione iniziale

$$x(0) = 0 \quad \rightarrow \quad x\dot{x}|_0 = 0 \quad \rightarrow \quad \left. \frac{d\langle x^2 \rangle}{dt} \right|_0 = 0$$

$$\frac{d\langle x^2 \rangle}{dt} = \frac{2k_B T}{\gamma} \left(1 - e^{-(\gamma/m)t} \right)$$



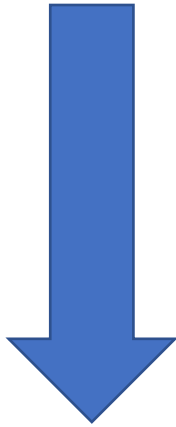
si integra ancora una volta ...

$$\langle x^2 \rangle = \frac{2k_B T}{\gamma} \left(t + \frac{m}{\gamma} e^{-(\gamma/m)t} - \frac{m}{\gamma} \right)$$

comportamenti diversi per tempi molto brevi o molto lunghi ...

$$\langle x^2 \rangle = \frac{2k_B T}{\gamma} \left(t + \frac{m}{\gamma} e^{-(\gamma/m)t} - \frac{m}{\gamma} \right)$$

$t \ll m/\gamma$



$t \gg m/\gamma$

$$\langle x^2 \rangle \rightarrow \frac{2k_B T}{\gamma} t$$

$$\langle x^2 \rangle \approx \frac{2k_B T}{\gamma} \left[t + \frac{m}{\gamma} \left(1 - \frac{\gamma}{m} t + \frac{\gamma^2}{2m^2} t^2 \right) - \frac{m}{\gamma} \right] = \frac{k_B T}{m} t^2$$

(ancora un'altra forma di teorema di fluttuazione-dissipazione)

Versione discreta del processo di Wiener (utile nelle simulazioni al computer)

$$x_{n+1} = x_n + w_n$$

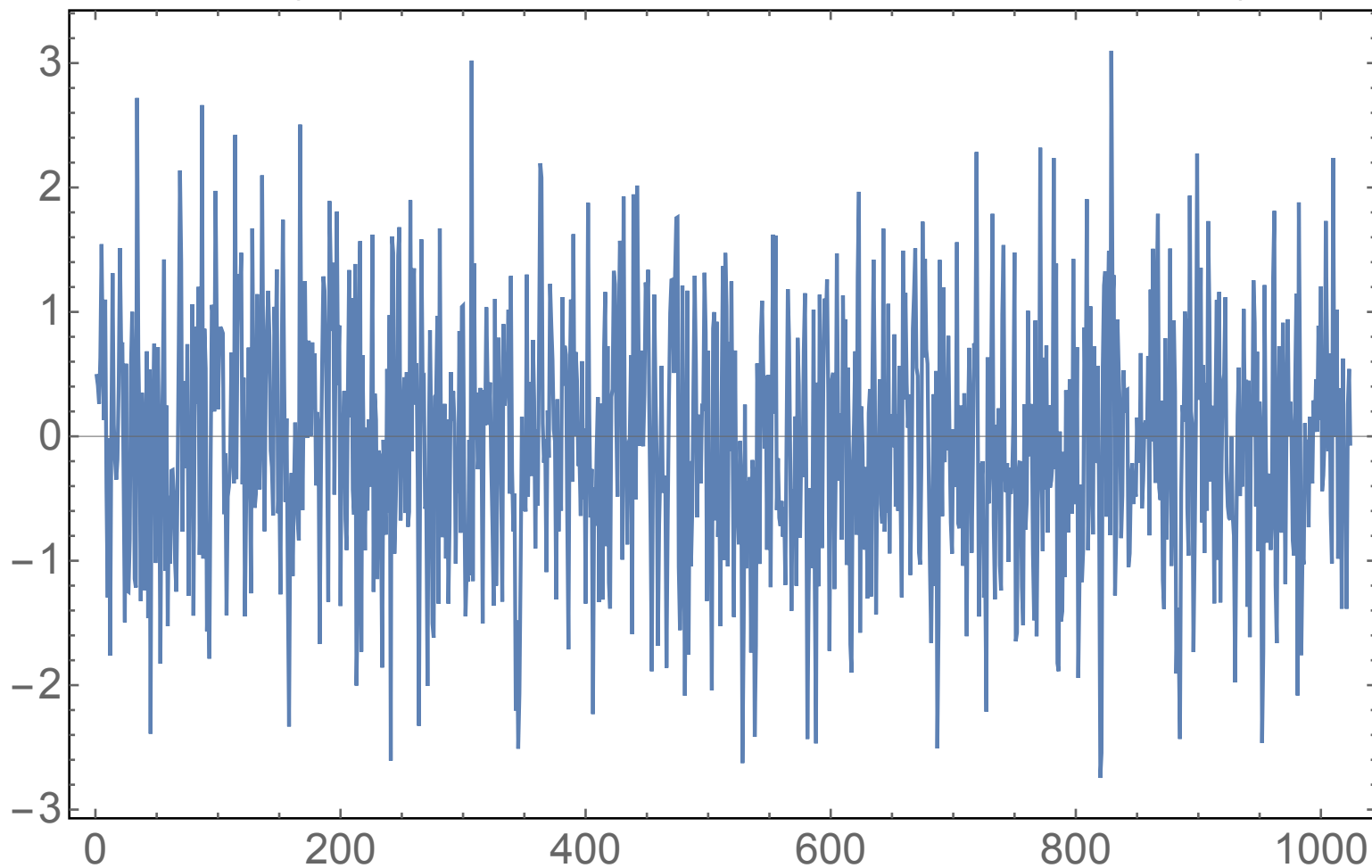


processo di Wiener

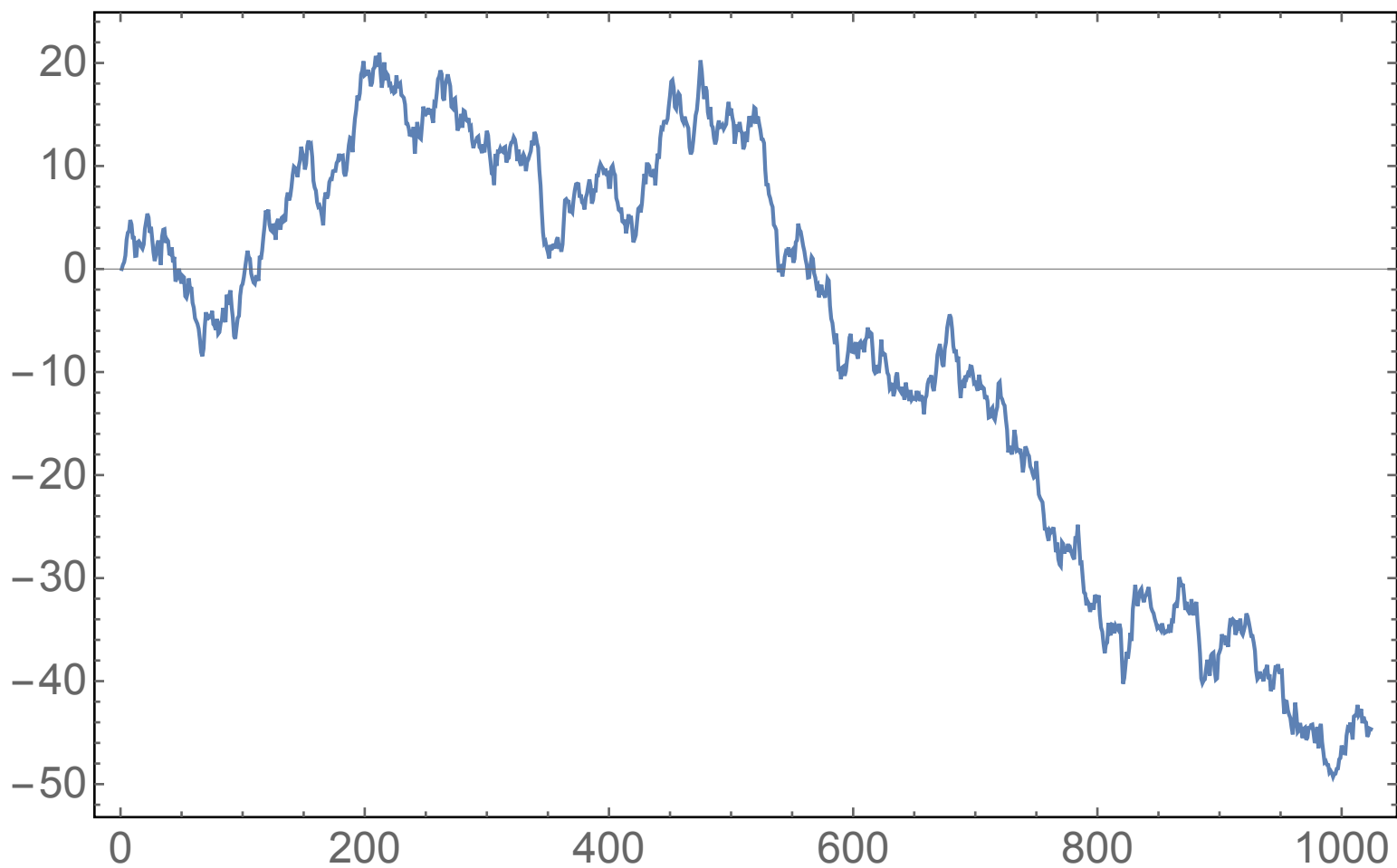


gli incrementi sono dati da
un rumore bianco
Gaussiano)

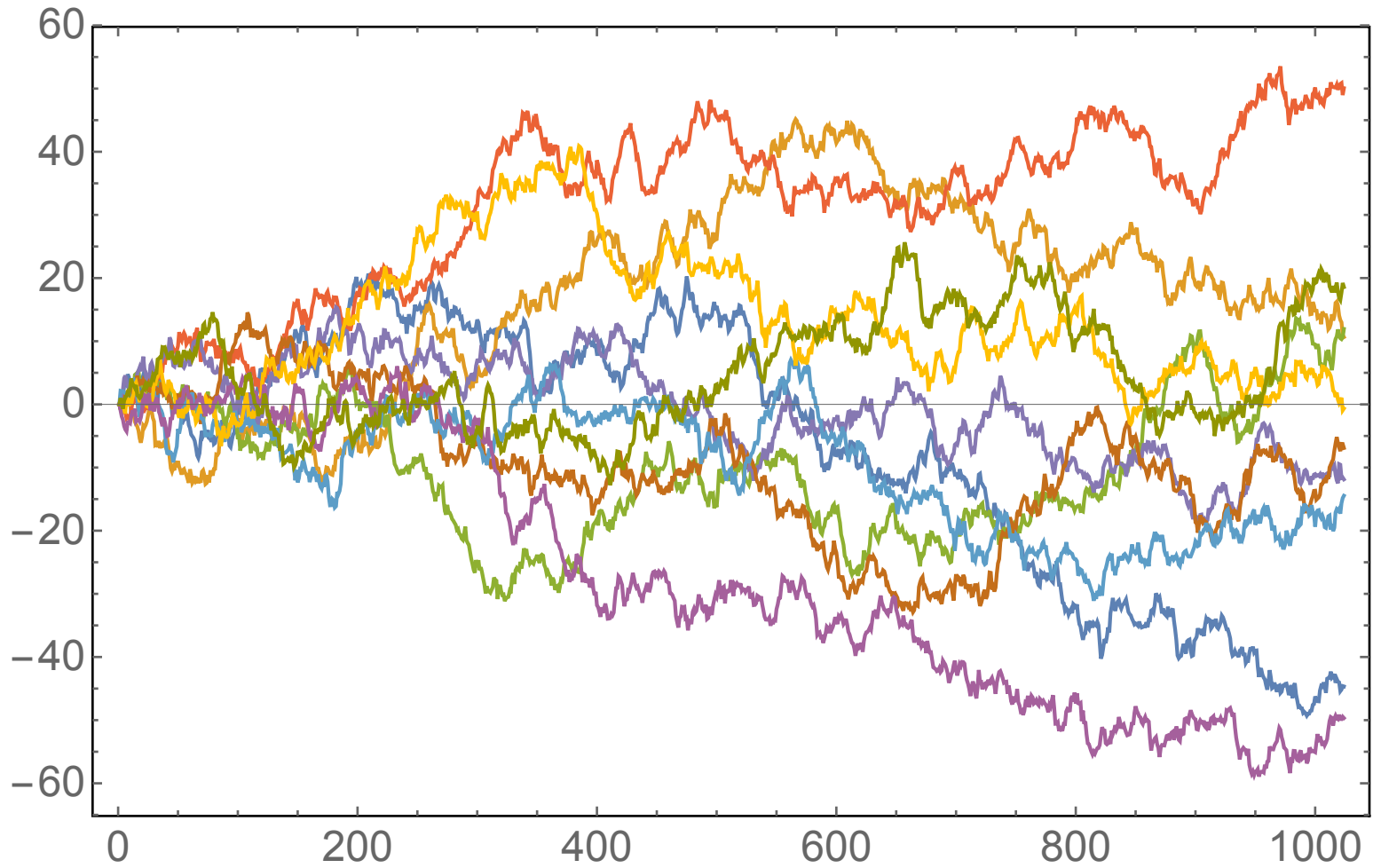
incrementi (una particolare realizzazione di un rumore bianco)



la corrispondente realizzazione 1-D del processo di Wiener



10 diverse realizzazioni del processo di Wiener



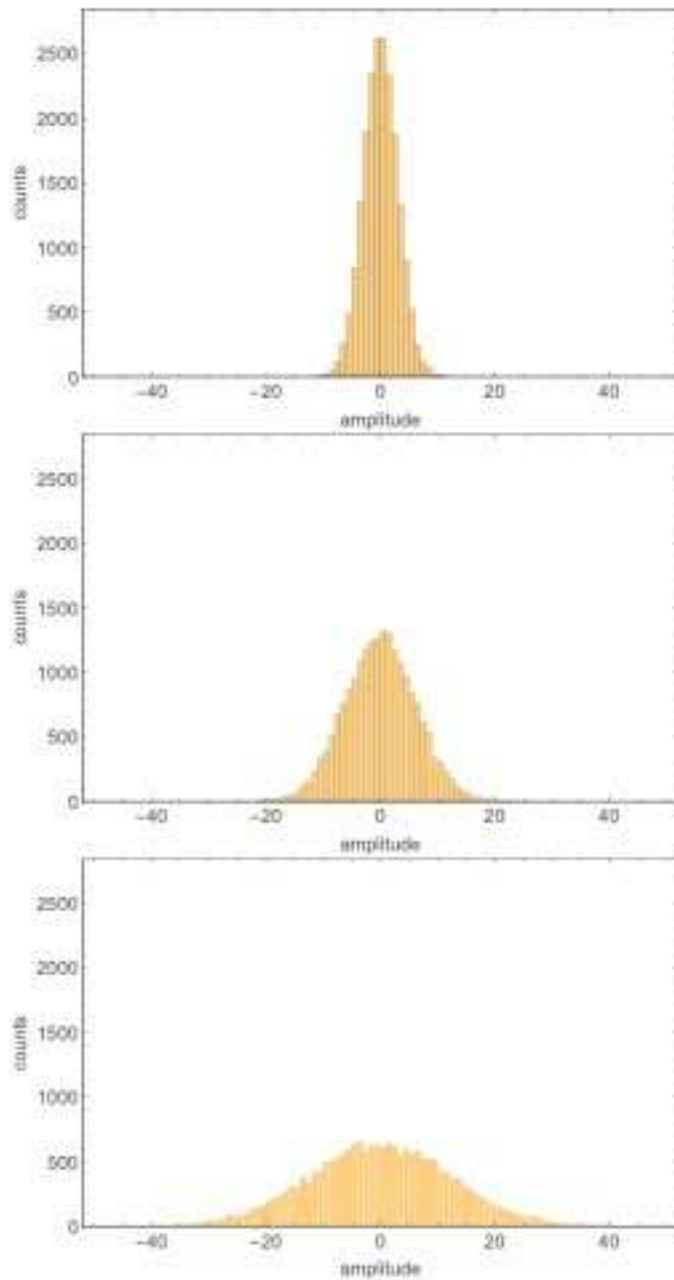


Figure 3.11: Amplitude histograms for a large ensemble of random walks (20000 walks). These snapshots correspond to different number of steps; top panel, 10 steps; middle panel, 40 steps; bottom panel, 160 steps. Although the number of steps is not very large, it can be shown that these histograms are very close to Gaussian distributions. So, the ensemble average is very different from the time average. This is an example of a process that is not ergodic in any finite time interval.

Densità spettrale del processo di Wiener

$$m \frac{dv}{dt} = -\gamma v(t) + \xi v(t) \quad \leftarrow \text{un "processo di Wiener"}$$
$$\frac{dx}{dt} = v(t) \quad \leftarrow \text{un "processo di Ornstein-Uhlenbeck"}$$

Si noti che il processo di Ornstein-Uhlenbeck si ottiene integrando un processo di Wiener.

Si noti ora che

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} d\omega \Rightarrow f'(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} (i\omega) F(\omega) e^{i\omega t} d\omega$$

$$\Rightarrow \mathcal{F}[f'(t)] = i\omega \mathcal{F}[f(t)]$$

e quindi la relazione tra densità spettrali è la seguente:

$$S_{f'}(\omega) = \omega^2 S_f(\omega)$$

o anche

$$S_f(\omega) = \frac{1}{\omega^2} S_{f'}(\omega)$$

Quindi la relazione tra densità spettrali di velocità e posizione è

$$S_v(\omega) = \omega^2 S_x(\omega)$$

Dall'equazione di Langevin si trova

$$m \frac{dv}{dt} = -\gamma v(t) + \xi v(t)$$

$$\Rightarrow m(i\omega)F_v(\omega) = -\gamma\omega F_v(\omega) + \xi \quad \Rightarrow \quad F_v(\omega) = \frac{\xi}{-i\omega + \gamma}$$

$$\Rightarrow S_v(\omega) = \frac{\xi^2}{\omega^2 + \gamma^2}$$

e infine

$$S_x(\omega) = \frac{1}{\omega^2} \frac{\xi^2}{\omega^2 + \gamma^2}$$

Si noti anche che

$$S_v(\omega) = \frac{\xi^2}{\omega^2 + \gamma^2} \rightarrow \frac{\xi^2}{\omega^2} \quad (\omega \gg \gamma)$$

e

$$S_x(\omega) \approx \frac{\xi^2}{\omega^4} \quad (\omega \gg \gamma)$$

(per frequenze abbastanza alte, la densità spettrale del rumore Browniano – quando questo viene caratterizzato come un processo di Wiener – si comporta come una legge di potenza).