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Drawing an elephant with four complex parameters

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NOTES AND DISCUSSIONS

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The falling raindrop, revisited

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I reconsider the problem of a raindrop falling through mist, collecting mass, and generalize it to allow an arbitrary power-law form for the accretion rate. I show that the coupled differential equations can be solved by the simple trick of temporarily eliminating time in favor of the raindrop's mass as the independent variable. © *2010 American Association of Physics Teachers.*
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A perennial homework exercise in differential-equation-based courses in Newtonian mechanics is the problem of a raindrop falling through mist and collecting mass.[1–4] If the rate of accretion is assumed to depend only on the raindrop's current mass (or radius) and not on its velocity, then the solution is fairly straightforward. If, by contrast, the rate of accretion is taken to depend on both the mass and the velocity, one is faced with a pair of coupled differential equations, and the trick for disentangling them can be surprisingly difficult to find—not only for the student but also for the instructor who has forgotten the method after some years' absence and must rediscover it (as I can testify from my recent experience).[5]

Here I would like to show that a very general version of the raindrop problem [see Eqs. (5) and (6)] can be solved by using a versatile technique that ought to have a place in all students' (and instructors') mathematical arsenals: namely, eliminating reference to the old independent variable (here the time t) and temporarily taking one of the former dependent variables as the new independent variable.

Students may remember this trick from the analysis of one-dimensional motion with a force that depends on position (x) and velocity (v) but not explicitly on time,

$$m\frac{dv}{dt} = F(x,v). \tag{1}$$

By using the chain rule

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}, \tag{2}$$

we can temporarily eliminate t and instead take x as the new independent variable: this elimination yields the first-order differential equation

$$mv\frac{dv}{dx} = F(x,v) \tag{3}$$

for the unknown function $v(x)$. In some cases Eq. (3) can be solved explicitly.[6] Once the function $v(x)$ is known, we can reinstate time and solve (in principle) the first-order separable differential equation

$$\frac{dx}{dt} = v(x) \tag{4}$$

to find the motion $x(t)$.

Let us now consider the raindrop problem, which involves a pair of coupled differential equations for two unknown functions: the raindrop's mass $m(t)$ and its velocity $v(t)$. The first equation is the Newtonian equation of motion for the raindrop,

$$m\frac{dv}{dt} + v\frac{dm}{dt} = mg, \tag{5}$$

which is obtained by the standard procedure of looking at the same collection of water particles (the "system") at two times, t and $t+\Delta t$, and writing that the rate of change of the system's total momentum equals the total external force on the system. The second equation states the hypothesized law of mass accretion for the raindrop. I shall consider the general form

$$\frac{dm}{dt} = \lambda m^{\alpha} v^{\beta}, \tag{6}$$

where $\lambda > 0$ is a constant and α and β are (almost) arbitrary exponents. This form includes the two most commonly studied cases, namely, the easy case $(\alpha,\beta)=(\frac{2}{3},0)$ [accretion proportional to the surface area of a spherical raindrop, with resulting acceleration $g/4$] and the difficult case $(\alpha,\beta)=(\frac{2}{3},1)$ [accretion proportional to the volume swept out, with resulting acceleration $g/7$] but is much more general.[7] I will show that all these cases can be solved by a unified technique.

First, a few preliminary remarks. Because Eqs. (5) and (6) are a pair of first-order differential equations for two unknown functions, the general solution will contain two constants of integration. Because this system is time-translation-invariant, one of the constants of integration simply sets the origin of time. The other constant of integration fixes the relation between the initial mass and the initial velocity: that is, it fixes the mass at the moment when the velocity has a specified value (or vice versa). The simplest solution arises

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by demanding that $m=0$ when $v=0$, but we will make some partial progress toward finding the general solution as well.

As mentioned, all the cases with $\beta=0$ are easy: the accretion equation (6) can be solved immediately for $m(t)$ [it is separable], and the Newtonian equation (5) can then be solved for $v(t)$ [it is linear first-order with nonconstant coefficients]. The trouble arises when $\beta \neq 0$ because now Eqs. (5) and (6) are coupled.

To decouple them, we employ the technique mentioned earlier. We use the chain rule in the form

$$\frac{dv}{dt} = \frac{dv}{dm}\frac{dm}{dt}, \tag{7}$$

forget temporarily about the time t, and instead consider the velocity v to be a function of the mass m (that is, we temporarily use m as the independent variable). Inserting Eq. (7) into Eq. (5) and using Eq. (6) to eliminate dm/dt (which now multiplies both terms on the left-hand side), we obtain

$$v^{\beta}\frac{dv}{dm} + \frac{v^{1+\beta}}{m} = \frac{g}{\lambda}m^{-\alpha}. \tag{8}$$

If we now make the change of variables $w=v^{1+\beta}$, we find[8]

$$\frac{dw}{dm} + \frac{1+\beta}{m}w = \frac{(1+\beta)g}{\lambda}m^{-\alpha}, \tag{9}$$

which is a first-order linear differential equation with nonconstant coefficients for the function $w(m)$. The integrating factor is $m^{1+\beta}$. After standard manipulations we obtain the general solution[9]

$$v = \left[\frac{(1+\beta)g}{(2+\beta-\alpha)\lambda}m^{1-\alpha} + \frac{C}{m^{1+\beta}}\right]^{1/(1+\beta)}. \tag{10}$$

It is convenient to express the constant of integration C in terms of the raindrop's mass m_0 at the moment its velocity is zero, which leads to[10]

$$v = Km^{(1-\alpha)/(1+\beta)}[1-(m_0/m)^{2+\beta-\alpha}]^{1/(1+\beta)}, \tag{11}$$

where $K=[(1+\beta)g/(2+\beta-\alpha)\lambda]^{1/(1+\beta)}$.

The simplest case is $m_0=0$. Then solving Eq. (11) for m, we have[11]

$$m = K'v^{(1+\beta)/(1-\alpha)}, \tag{12}$$

where K' is a constant that we need not write out explicitly. We now insert Eq. (12) into the Newtonian equation (5). Because each of the terms in this equation is linear in m, the constant K' drops out, and we obtain

$$\frac{dv}{dt} = \frac{1-\alpha}{2+\beta-\alpha}g. \tag{13}$$

We have thus shown that the raindrop falls with constant acceleration $[(1-\alpha)/(2+\beta-\alpha)]g$. [When $(\alpha,\beta)=(\frac{2}{3},0)$ or $(\frac{2}{3},1)$, Eq. (13) gives the acceleration $g/4$ or $g/7$, respectively.] To obtain $m(t)$, it suffices to observe that v is proportional to t (if we take the origin of time to be the instant when $v=0$), so that from Eq. (12) we obtain $m(t)=K''t^{(1+\beta)/(1-\alpha)}$.

If $m_0 \neq 0$, the best approach seems to be to insert Eq. (11) into Eq. (6) and integrate,

$$\int m^{-[(\alpha+\beta)/(1+\beta)]}[1-(m_0/m)^{2+\beta-\alpha}]^{-[\beta/(1+\beta)]}\,dm = \lambda K^{\beta}\int dt. \tag{14}$$

If $\beta=0$, the integral is easy, and we obtain

$$m(t) = m_0[1+(1-\alpha)\lambda t/m_0^{1-\alpha}]^{1/(1-\alpha)}, \tag{15}$$

and

$$v(t) = \frac{1-\alpha}{2-\alpha}gt + \frac{m_0^{1-\alpha}g}{(2-\alpha)\lambda} \times\{1-[1+(1-\alpha)\lambda t/m_0^{1-\alpha}]^{-[1/(1-\alpha)]}\}. \tag{16}$$

If $\beta \neq 0$, the substitution $z=(m_0/m)^{2+\beta-\alpha}$ allows the left-hand side of Eq. (14) to be expressed in terms of the incomplete beta function

$$B(z;a,b) = \int_0^z t^{a-1}(1-t)^{b-1}dt, \tag{17}$$

namely,

$$\lambda K^{\beta}t = -\frac{1}{2+\beta-\alpha}m_0^{(1-\alpha)/(1+\beta)}B((m_0/m)^{2+\beta-\alpha};a,b) \tag{18}$$

with $a=-(1-\alpha)/[(1+\beta)(2+\beta-\alpha)]$ and $b=1/(1+\beta)$. (Equation (18) can alternatively be written in terms of a hypergeometric function ${}_2F_1$ if one prefers.) It seems difficult to make further analytic progress. We can in any case see from Eq. (14) that the long-time behavior is

$$m(t) = K''t^{(1+\beta)/(1-\alpha)}\left(1+\sum_{k=1}^{\infty}a_k t^{-k(1+\beta)(2+\beta-\alpha)/(1-\alpha)}\right), \tag{19}$$

and hence

$$v(t) = \frac{1-\alpha}{2+\beta-\alpha}gt\left(1+\sum_{k=1}^{\infty}b_k t^{-k(1+\beta)(2+\beta-\alpha)/(1-\alpha)}\right), \tag{20}$$

from which the coefficients $\{a_k\}$ and $\{b_k\}$ can be determined by substitution into Eqs. (5) and (6).

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[5] As a recent article put it, "undergraduate mechanics students are *sometimes* [my emphasis] able to solve the nonlinear dynamical equations of motion to find the deceptively simple acceleration $g/7$" [B. F. Edwards, J. W. Wilder, and E. E. Scime, "Dynamics of falling raindrops," Eur. J. Phys. **22**, 113–118 (2001)]. A physics student, answering another student's query on an online forum, was blunter: "This is a very old problem. Unfortunately, I remember the answer, $g/7$, but I don't remember how you get it. It has an unusual solution. There [is] a special substitution that you need to make for the mass; otherwise the problem is insoluble" ⟨www.physicsforums.com/showthread.php?t=198859⟩.

[6] For instance, if $F(x,v)=g(x)h(v)$, then Eq. (3) is separable [in particular, when $F(x,v)=g(x)$, the solution to Eq. (3) gives the usual conservation-of-energy equation]. If $F(x,v)=g(x)v+h(x)v^2$, then Eq. (3) is a first-order linear equation with nonconstant coefficients for the function $v(x)$; and more generally, if $F(x,v)=g(x)v^{2-\alpha}+h(x)v^2$, then Eq. (3) is a first-order linear equation for the function $v(x)^\alpha$. Likewise, if $F(x,v)=v/[g(v)+h(v)x]$, then Eq. (3) can be turned upside-down to obtain a first-order linear equation with nonconstant coefficients for the inverse function $x(v)$; and more generally, if $F(x,v)=v/[g(v)x^{1-\beta}+h(v)x]$, then Eq. (3) gives a first-order linear equation for the function $x(v)^\beta$.

[7] At least two other cases of Eq. (6) correspond to physically realizable (albeit highly artificial) situations: namely, $(\alpha,\beta)=(0,1)$ and $(1,1)$ arise when the raindrop is constrained (for example, by a massless container) to be a cylinder of fixed base and growing height (respectively, fixed height and growing base). Note that the cylinder's base can be of arbitrary shape and need not be circular.

[8] The case $\beta=-1$ needs to be treated separately and yields $v=Cm^{-1}\exp[\{g/[(1-\alpha)\lambda]\}m^{1-\alpha}]$. Of course, all cases $\beta<0$ are unphysical.

[9] The case $\alpha=2+\beta$ needs to be treated separately and yields $v=(\{[(1+\beta)g]/\lambda\}\{[\log(m/m_0)]/m^{1+\beta}\})^{1/(1+\beta)}$. Of course, all cases $\alpha>1$ are probably unphysical.

[10] This step assumes $\beta>-1$.

[11] This step assumes $\beta>-1$ and $\alpha<1$.

Comment on "Generalized composition law from 2×2 matrices," by R. Giust, J.-M. Vigoureux, and J. Lages [Am. J. Phys. 77 (11), 1068–1073 (2009)]

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In a recent paper, Giust *et al.*[1] showed how the elements of the 2×2 scattering matrix of a one-dimensional system composed of two subsystems can be expressed in terms of the scattering amplitudes of the individual subsystems to generalize Einstein's composition law of velocities.

We note that this factorization property of the one-dimensional S-matrix (Eq. (21) in Ref. 1) was first derived by Aktosun[2] for the one-dimensional Schrödinger equation and for the wave equation in nonhomogeneous, nondispersive mediums when the wave speed has the same asymptotic behavior at both ends of the real line. Aktosun presented this result in a very compact form using Λ-matrices that can be associated with the S-matrices. We write the on-shell S-matrix as

$$S=\begin{pmatrix} T & R \\ L & T \end{pmatrix}, \tag{1}$$

with T as the transmission amplitude and L and R as the reflection amplitudes from the left and right, respectively. We can associate with S the Λ-matrix,

$$\Lambda=\begin{pmatrix} \dfrac{1}{T} & -\dfrac{R}{T} \\[2ex] \dfrac{L}{T} & \dfrac{1}{T^*} \end{pmatrix}. \tag{2}$$

If the system is composed of two subsystems (or fragments; see Ref. 2 for details), characterized by the S-matrices S_1 and S_2 and the corresponding Λ-matrices Λ_1 and Λ_2, the following factorization property can be proven:

$$\Lambda=\Lambda_1\Lambda_2. \tag{3}$$

In other terms, the composition of subsystems turns into simple multiplication of the Λ-matrices associated with them. For a system made of N subsystems, this property can be iterated to yield the more general result,[2]

$$\Lambda=\Lambda_1\cdots\Lambda_N. \tag{4}$$

Following the original proof given by Aktosun,[2] the factorization formula was derived by different methods in different contexts. Together with Klaus and van der Mee, Aktosun generalized the proof to allow for the inclusion of Dirac delta functions in the potential.[3] Sassoli de Bianchi and Di Ventra derived an expression for a position-dependent mass using an adaptation of the variable phase method, which also allowed them to derive first-order differential equations for the transmission and reflection amplitudes, which are particularly convenient for numerical computation.[4] An alternative proof using integral equations instead of the Schrödinger equation was proposed in Ref. 5. Also, Sprung, Wu and Martorell[6] derived a factorization formula in the context of finite periodic potentials, emphasizing the crucial role played by the Λ-matrix (called M in their work) associated with the single periodically repeated subsystem.

We also comment on the utility of the factorization formula in relation to phase variables, as emphasized in Ref. 1 by several examples. Another important example is the possibility of using the factorization formula in combination with Levinson's theorem to characterize the number of bound states of a system. Levinson's theorem is a classic result, which establishes a simple relation between the num-

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ber of bound states and the behavior of its scattering phase-shifts at zero energy. In one dimension the theorem can be expressed using the phase $\alpha = \arg T$ of the transmission amplitude.[7] Because of the factorization property, the transmission phase can be expressed as the sum of the transmission phases of the individual subsystems, plus an overall interference term,

$$\alpha = \alpha_1 + \cdots + \alpha_N + A. \quad (5)$$

By studying the zero-energy limit of the interference contribution A, it becomes possible, using Levinson's theorem, to characterize the number of bound states of the composite system as a function of the number of bound states of its subsystems. Interestingly, the problem admits a general and explicit solution for finite periodic structures, allowing for a complete characterization of the number of states bound by a superlattice as a function of the spacing between the individual cells.[8,9]

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Comment on "How fast could Usain Bolt have run? A dynamical study" by H. K. Eriksen, J. R. Kristiansen, Ø. Langangen, and I. K. Wehus [Am. J. Phys. 77(3), 224–228 (2009)]

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In this paper[1] the authors asserted that if Usain Bolt had not celebrated during the last 2 s of his historic Beijing 2008 100 m run, he could have run a 9.61 ± 0.04 s time if he kept up with the runner up's acceleration or a time of 9.55 ± 0.04 s if he had an acceleration greater than that of the runner up by 0.5 $\mathrm{m/s^2}$. These assertions are based on measurements extracted from video and photo analysis to obtain split times of the race, which in turn are utilized to extrapolate, by integrating kinematic equations, the last 2 s of the race for the two scenarios.

First and foremost the authors should be commended on obtaining a meaningful statistical analysis and incorporating an amusing application of physics to the world of sports. Their main concern was not to provide an absolute prediction to be utilized by 100 m coaches and enthusiasts but more of a pedagogical device to motivate students even further in physics. They had at their disposal very crude web footage, and thus one of their main concerns was that their error estimates were correct.

It should be noted though that if the authors had access to very precise 10 m split times (times recorded at each 10 m interval), then the calculation of the maximum speed that Usain Bolt would have run is simply accomplished by observing the distance that he obtained his maximum velocity (usually for a male world class sprinter around the 60 m or 70 m mark) and assume that this speed is held constant throughout the remainder of the race. This will be the fastest time possible, but it is highly improbable.

Obviously, sprinters do not accelerate after they achieved their maximum velocity and toward the end of the race they reach the part of the race called the deceleration phase.[2] It is hard to believe but sprinters actually slow down at the end of the 100 m race even though their effort is 100%.[3] Often one hears that runners "kick it into a higher gear" if they happen to pass up a sprinter in the last 20 or 10 m of the 100 m race. It is basically an optical illusion and what really occurs is that the faster runner's rate of deceleration is simply less than the competitor.

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[1]H. K. Eriksen, J. R. Kristiansen, Ø. Langangen, and I. K. Wehus, Am. J. Phys. **77**(3), 224–228 (2009).
[2]For a concise and very informative explanation visit http://www.iaaf.org/community/athletics/trackfield/newsid=4666.html
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Comment on "Resistance of a square lamina by the method of images," by Lawrence R. Mead [Am. J. Phys. 77 (3), 259–261 (2009)]

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Lawrence R. Mead computed the resistance of square lamina with "point contacts" at two opposite corners.[1] To this end, he used an infinite number of image charges. The use of image charges to meet boundary conditions is a common method in electrostatics. However, the problem to be solved is not an electrostatics problem as it involves resistance and thus current (i.e., nonstatic charge).

The potential around a current source or sink in a lamina is not the same as that around an electrostatic charge. To derive the potential around a current source, we consider a single current source at the origin of the coordinate system in a lamina extending to infinity. The current will flow in all directions, leading to circular equipotential lines around the source. The sum of all current sources minus the sum of all current sinks within a certain region gives the net flow out of the region. Because we only have one single source, the current through an arbitrary contour enclosing the source should always be equal to the current coming out of the source. The current-density and the electric field must then be inversely dependent on the distance from the source, we write for the electric field

$$\boldsymbol{E}(\boldsymbol{r}) = R_s \boldsymbol{J}(\boldsymbol{r}) = \frac{qR_s\boldsymbol{r}}{2\pi|\boldsymbol{r}|^2}, \tag{1}$$

where $\boldsymbol{r}$ is a vector pointing from the source, $\boldsymbol{J}$ is the current density, q is the strength of the source (i.e., the current flowing out of the source), and R_s is the sheet resistance. Integrating this electric field we obtain the potential

$$V(\boldsymbol{r}) = -\int_{r_c}^{|\boldsymbol{r}|} \frac{qR_s}{2\pi r'} dr' = \frac{qR_s}{2\pi} \ln\left(\frac{r_c}{|\boldsymbol{r}|}\right), \tag{2}$$

where r_c is the radius of the source (contact). A similar approach to the one presented by Mead can now be used to evaluate the resistance but with Eq. (2) used instead of the electrostatic potential of a point charge.

Note that the divergence of the electric field in Eq. (1) is 0 everywhere except at the origin. This reflects the fact that there are no sources or sinks except for the source at the origin. The divergence of the electric field of an electrostatic charge, however, is nonzero everywhere. As a result the expression for the current found by Mead [Eq. (12) in his paper] depends on the radius of the arc enclosing the contact. However, the current should not depend on the radius in the absence of additional sources or sinks enclosed by the arc.

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[1] L. R. Mead, "Resistance of a square lamina by the method of images," Am. J. Phys. **77**(3), 259–261 (2009).

Reply to "Comment on 'Resistance of a square lamina by the method of images,'" by B. E. Pieters [Am. J. Phys. 78 (6), 647–647 (2010)]

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I would like to thank B. E. Pieters for his comment.[1] While the *purely electrostatic* boundary conditions are satisfied by charges whose fields are radial in *either* two or three dimensions (just vector addition), the correct mapping of the electrostatic problem back to the original current problem requires adopting Eq. (1) in Pieters' comment for the field in a plane. There, $\vec{r}$ is the cylindrical radius vector. Using that field, whose divergence vanishes, we have the additional constraint that the current must be independent of the radius of the arc enclosing the contact, which is a somewhat subtle point that I missed in my original paper.[2]

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[1] B. E. Pieters, "Comment on 'Resistance of a square lamina by the method of images,' by L. R. Mead [Am. J. Phys. **77**(3), 259–261 (2009)]," Am. J. Phys. **78**(6), 647–647 (2010).
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Drawing an elephant with four complex parameters

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We define four complex numbers representing the parameters needed to specify an elephantine shape. The real and imaginary parts of these complex numbers are the coefficients of a Fourier coordinate expansion, a powerful tool for reducing the data required to define shapes. © *2010 American Association of Physics Teachers.*
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A turning point in Freeman Dyson's life occurred during a meeting in the Spring of 1953 when Enrico Fermi criticized the complexity of Dyson's model by quoting Johnny von Neumann:[1] "With four parameters I can fit an elephant, and with five I can make him wiggle his trunk." Since then it has become a well-known saying among physicists, but nobody has successfully implemented it.

To parametrize an elephant, we note that its perimeter can be described as a set of points $(x(t), y(t))$, where t is a parameter that can be interpreted as the elapsed time while going along the path of the contour. If the speed is uniform, t becomes the arc length. We expand x and y separately[2] as a Fourier series

$$x(t) = \sum_{k=0}^{\infty} (A_k^x \cos(kt) + B_k^x \sin(kt)), \tag{1}$$

$$y(t) = \sum_{k=0}^{\infty} (A_k^y \cos(kt) + B_k^y \sin(kt)), \tag{2}$$

where A_k^x, B_k^x, A_k^y, and B_k^y are the expansion coefficients. The lower indices k apply to the kth term in the expansion, and the upper indices denote the x or y expansion, respectively.

Using this expansion of the x and y coordinates, we can analyze shapes by tracing the boundary and calculating the coefficients in the expansions (using standard methods from Fourier analysis). By truncating the expansion, the shape is smoothed. Truncation leads to a huge reduction in the information necessary to express a certain shape compared to a pixelated image, for example. Székely *et al.*[3] used this approach to segment magnetic resonance imaging data. A similar approach was used to analyze the shapes of red blood cells,[4] with a spherical harmonics expansion serving as a 3D generalization of the Fourier coordinate expansion.

The coefficients represent the best fit to the given shape in the following sense. The $k=0$ component corresponds to the center of mass of the perimeter. The $k=1$ component corresponds to the best fit ellipse. The higher order components trace out elliptical corrections analogous to Ptolemy's epicycles.[5] Visualization of the corresponding ellipses can be found at Ref. 6.

We now use this tool to fit an elephant with four parameters. Wei[7] tried this task in 1975 using a least-squares Fourier sine series but required about 30 terms. By analyzing the picture in Fig. 1(a) and eliminating components with amplitudes less than 10% of the maximum amplitude, we obtained an approximate spectrum. The remaining amplitudes were

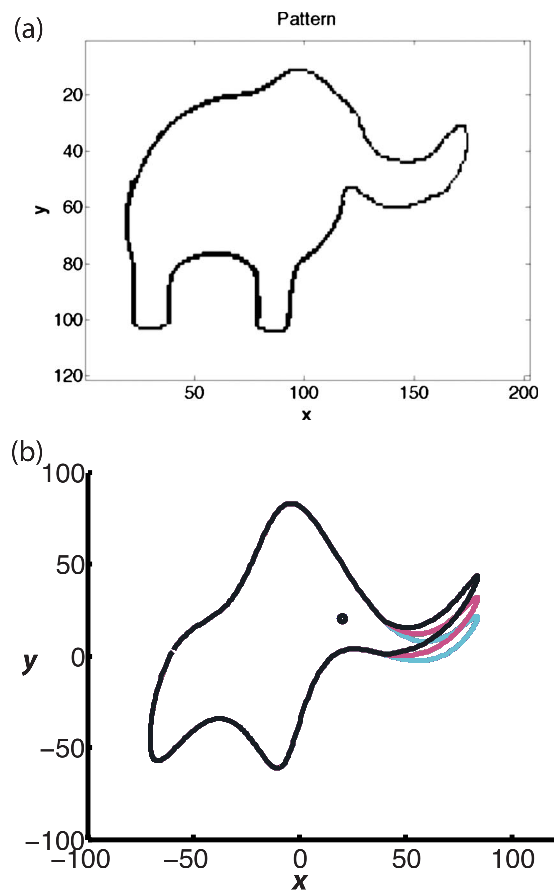

Fig. 1. (a) Outline of an elephant. (b) Three snapshots of the wiggling trunk.

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Table I. The five complex parameters $p_1, \ldots, p_5$ that encode the elephant including its wiggling trunk.

Parameter	Real part	Imaginary part
$p_1=50-30i$	$B_1^x=50$	$B_1^y=-30$
$p_2=18+8i$	$B_2^x=18$	$B_2^y=8$
$p_3=12-10i$	$A_3^x=12$	$B_3^y=-10$
$p_4=-14-60i$	$A_5^x=-14$	$A_1^y=-60$
$p_5=40+20i$	Wiggle coeff. $=40$	$x_{\mathrm{eye}}=y_{\mathrm{eye}}=20$

slightly modified to improve the aesthetics of the final image. By incorporating these coefficients into complex numbers, we have the equivalent of an elephant contour coded in a set of four complex parameters (see Fig. 1(b)).

The real part of the fifth parameter is the "wiggle parameter," which determines the x-value where the trunk is attached to the body (see the video in Ref. 8). Its imaginary part is used to make the shape more animal-like by fixing the coordinates for the elephant's eye. All the parameters are specified in Table I.

The resulting shape is schematic and cartoonlike but is still recognizable as an elephant. Although the use of the Fourier coordinate expansion is not new,[2,3] our approach clearly demonstrates its usefulness in reducing the number of parameters needed to describe a two-dimensional contour. In the special case of fitting an elephant, it is even possible to lower it to four complex parameters and therein implement a well-known saying.

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