

SIGNAL DISCOVERY, LIMITS, AND UNCERTAINTIES WITH SPARSE ON/OFF MEASUREMENTS: AN OBJECTIVE BAYESIAN ANALYSIS

MAX L. KNOETIG

Institute for Particle Physics, ETH Zurich, 8093 Zurich, Switzerland; mknoetig@phys.ethz.ch
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ABSTRACT

For decades researchers have studied the On/Off counting problem where a measured rate consists of two parts. One part is due to a signal process and the other is due to a background process, the magnitudes for both of which are unknown. While most frequentist methods are adequate for large number counts, they cannot be applied to sparse data. Here, I want to present a new objective Bayesian solution that only depends on three parameters: the number of events in the signal region, the number of events in the background region, and the ratio of the exposure for both regions. First, the probability of the counts only being due to background is derived analytically. Second, the marginalized posterior for the signal parameter is also derived analytically. With this two-step approach it is easy to calculate the signal's significance, strength, uncertainty, or upper limit in a unified way. This approach is valid without restrictions for any number count, including zero, and may be widely applied in particle physics, cosmic-ray physics, and high-energy astrophysics. In order to demonstrate the performance of this approach, I apply the method to gamma-ray burst data.

Key words: gamma rays: general – methods: statistical

Online-only material: color figure

1. INTRODUCTION

Typical counting experiments measure a discrete set of events, such as the decay time of a particle. Such data are often (Li & Ma 1983; Cousins et al. 2008) modeled with the Poisson distribution given by

$$P_P(N|\lambda) = \frac{e^{-\lambda}\lambda^N}{N!}. \quad (1)$$

This distribution connects the probability of observing N events to a nonnegative number of expected events, λ , derived, for example, from a rate in a fixed time interval or a luminosity and a cross section. The Poisson distribution may be approximated by a Gaussian distribution when measuring many events. However, when the data are sparse, such an approximation is not good enough. Indeed, in some areas, this is often the rule rather than the exception, such as in high-energy astrophysics (Loredo 1992). In this paper, a full Bayesian analysis of On/Off data, valid for any count number, is presented.

2. THE ON/OFF MEASUREMENT

In the On/Off problem, one would like to infer a signal rate in the presence of an imprecisely known background. The measurement consists of the observation of N_{off} events in a region chosen a priori to be signal free and the observation of N_{on} events in the region of a potential signal in addition to the background.

The notation comes from astronomy, where telescopes point on and off potential source regions. In particle physics, the off region is taken in some region close to the signal region in the measured parameter (typically called sideband; e.g., Cousins et al. 2008) or without a radioactive signal source near the detector.

The ratio of the exposure for both regions α is assumed to be known with negligible uncertainty. In gamma-ray astronomy, in the simplest cases α is

$$\alpha = \frac{A_{\text{on}}t_{\text{on}}}{A_{\text{off}}t_{\text{off}}}, \quad (2)$$

where A stands for the size and t for the exposure time of the regions. Berge et al. (2007) illustrate how to generalize Equation (2) for complex acceptances. Given N_{on} , N_{off} , and α , the problem is then to calculate the evidence for a signal and the posterior distribution of the signal parameter.

Frequentist analyses, based on likelihood ratios and other methods, are widely used in particle physics (Cousins et al. 2008) and in high-energy astrophysics (as promoted by Li & Ma 1983). However, they often assume normally distributed random numbers and therefore lose their foundation when applied to low count numbers.

Gillessen & Harney (2005) have proposed a Bayesian solution to the question of whether or not the signal parameter is larger than zero. However, they do not account for the alternative, simpler hypothesis asserting that all N_{on} come from background only, i.e., no source. Therefore, the result is an overestimation, as pointed out by Gregory (2005). There exists a Bayesian solution to compute the odds ratio of the two hypotheses (Gregory 2005), but it adds an arbitrary parameter to the problem, depending on the prior (in particular its upper boundary), which makes the probability statement hard to interpret.

3. ANALYSIS

In this paper, I present an objective Bayesian analysis. This is accomplished by using improper priors as a tool to produce proper posteriors representing our lack of knowledge. Subjective Bayesian methods can certainly have benefits, in particular, when a prior opinion is strongly held. One can gain sensitivity by using informative priors that precisely specify prior knowledge, such as a prior source detection or a known background. However, objective Bayesian methods should be used when one is interested in “what the data have to say” (Irony & Singpurwalla 1997).

The analysis presented here follows the method outlined by Caldwell & Kröniger (2006) and may be considered an analytical special case. Agostini et al. (2013) applied the method in order to analyze and set stringent upper limits on the neutrino-less double-beta decay of ^{76}Ge . Kashyap et al. (2010) recently presented a similar frequentist method. The analysis is performed in two steps. First, the probability that the observed counts are due to background only is calculated. If this is lower than a previously defined consensus value, then the signal is said to be detected. Second, the signal contribution is estimated or an upper limit for the signal is calculated, depending on whether the detection limit has been reached.

3.1. Hypothesis Test

Let H_0 denote the null hypothesis that the observed counts are due to background only. The alternative hypothesis H_1 is that a signal process contributes to the counts. H_1 could be a bad model too, in case of systematic uncertainties. I should note that this is sometimes the case when, e.g., signal counts leak into the off region. Nevertheless, in the following, it is assumed that the systematic uncertainties are negligible and the two-model set of exclusive rival hypotheses $\{H_0, H_1\}$ is complete. By using Bayes' theorem, one may calculate the conditional probability of H_0 as

$$P(H_0|N_{\text{on}}, N_{\text{off}}) = \frac{P(N_{\text{on}}, N_{\text{off}}|H_0)P_0(H_0)}{P(N_{\text{on}}, N_{\text{off}})}, \quad (3)$$

where $P(N_{\text{on}}, N_{\text{off}}|H_i)$ is the conditional probability of observing the data, given the hypothesis H_i and that $P_0(H_i)$ is the prior probability for H_i . For a set of exclusive rival hypotheses such that $\sum_i P(H_i) = 1$ and $P(H_i \wedge H_j) = 0$ for $i \neq j$, the law of total probability gives

$$P(N_{\text{on}}, N_{\text{off}}) = \sum_i P(N_{\text{on}}, N_{\text{off}}|H_i)P_0(H_i). \quad (4)$$

Furthermore, in continuously parameterized models, the continuous counterpart of the law of total probability, with sums replaced by integrals, gives

$$P(N_{\text{on}}, N_{\text{off}}) = \sum_i \int P(N_{\text{on}}, N_{\text{off}}|\lambda_i, H_i)P_0(\lambda_i|H_i)d\lambda_i P_0(H_i). \quad (5)$$

One obtains the sum over the full set of hypotheses H_i and integrates with respect to their parameters λ_i . By assuming the two-hypothesis set $\{H_0, H_1\}$, one can write Equation (5) in terms of the expected number of signal events λ_s and the expected number of background events λ_{bg} :

$$P(N_{\text{on}}, N_{\text{off}}) = \int P(N_{\text{on}}, N_{\text{off}}|\lambda_{\text{bg}}, H_0)P_0(\lambda_{\text{bg}}|H_0)d\lambda_{\text{bg}}P_0(H_0) + \int P(N_{\text{on}}, N_{\text{off}}|\lambda_s, \lambda_{\text{bg}}, H_1)P_0(\lambda_s, \lambda_{\text{bg}}|H_1)d\lambda_s d\lambda_{\text{bg}}P_0(H_1). \quad (6)$$

Here, $P(N_{\text{on}}, N_{\text{off}}|\lambda_{\text{bg}}, H_0)$ and $P(N_{\text{on}}, N_{\text{off}}|\lambda_s, \lambda_{\text{bg}}, H_1)$ denote the conditional probabilities to measure the data.

Assuming that the number of signal events (if any) and the number of background events are independent, Poisson-distributed random variables with means λ_s and λ_{bg} , the expected number of events in the *off* region is

$$E(N_{\text{off}}) = \lambda_{\text{bg}}. \quad (7)$$

The expected number of events $E(N)$ in the *on* region, assuming the null hypothesis H_0 , is

$$E(N_{\text{on}}) = \alpha\lambda_{\text{bg}}, \quad (8)$$

or assuming H_1 , is

$$E(N_{\text{on}}) = \lambda_s + \alpha\lambda_{\text{bg}}. \quad (9)$$

For the conditional probabilities to measure the data or *likelihoods*, this yields

$$P(N_{\text{on}}, N_{\text{off}}|\lambda_{\text{bg}}, H_0) = P_{\text{P}}(N_{\text{on}}|\alpha\lambda_{\text{bg}})P_{\text{P}}(N_{\text{off}}|\lambda_{\text{bg}}), \quad (10)$$

and

$$P(N_{\text{on}}, N_{\text{off}}|\lambda_s, \lambda_{\text{bg}}, H_1) = P_{\text{P}}(N_{\text{on}}|\lambda_s + \alpha\lambda_{\text{bg}})P_{\text{P}}(N_{\text{off}}|\lambda_{\text{bg}}). \quad (11)$$

The priors $P_0(\lambda_{\text{bg}}|H_0)$ and $P_0(\lambda_s, \lambda_{\text{bg}}|H_1)$ are chosen according to *Jeffreys's rule* (see Jeffreys 1998; Beringer et al. 2012):

$$P_0(\lambda_i|H_i) \propto \sqrt{\det[I(\lambda_i|H_i)]}, \quad (12)$$

$$I_{kl}(\lambda_i|H_i) = -E \left[\frac{\partial^2 \ln L(N_{\text{on}}, N_{\text{off}}|\lambda_i, H_i)}{\partial \lambda_k \partial \lambda_l} \right], \quad (13)$$

where I_{kl} denotes the *Fisher information matrix*, L is the likelihood function (either Equation (10) or (11)), E is the expectation value with respect to the model with index i , and λ_i is its parameter vector. In Appendix A, I show that

$$P_0(\lambda_{\text{bg}}|H_0) \propto \sqrt{\frac{1+\alpha}{\lambda_{\text{bg}}}}, \quad (14)$$

$$P_0(\lambda_s, \lambda_{\text{bg}}|H_1) \propto \sqrt{\frac{1}{\lambda_{\text{bg}}(\alpha\lambda_{\text{bg}} + \lambda_s)}}. \quad (15)$$

The On/Off Jeffreys's priors are improper, i.e., they integrate to infinity over the parameter space. There is a debate among statisticians concerning the use of improper priors in Bayesian model selection (Berger & Pericchi 2001), as the priors are only specified up to the proportionality constants c_0, c_1 , which do not cancel out. The probability of H_0 to be true given the measured counts is therefore

$$P(H_0|N_{\text{on}}, N_{\text{off}}) = \frac{c_0\gamma'}{c_0\gamma' + c_1\delta'} \quad (16)$$

$$= \frac{\gamma'}{\gamma' + c_1/c_0\delta'} \quad (17)$$

with

$$\gamma' := \int_0^\infty P(N_{\text{on}}, N_{\text{off}}|\lambda_{\text{bg}}, H_0)P_0(\lambda_{\text{bg}}|H_0)d\lambda_{\text{bg}}P_0(H_0), \quad (18)$$

$$\delta' := \int_0^\infty \int_0^\infty P(N_{\text{on}}, N_{\text{off}}|\lambda_s, \lambda_{\text{bg}}, H_1) \times P_0(\lambda_s, \lambda_{\text{bg}}|H_1)d\lambda_s d\lambda_{\text{bg}}P_0(H_1). \quad (19)$$

To calculate the analytic outcome of Equation (17), the priors for the hypotheses $P_0(H_0)$ and $P_0(H_1)$ have to be identified.

Given the lack of prior information as to which hypothesis is more likely, they are chosen to be equal

$$P_0(H_0) = P_0(H_1) = \frac{1}{2}. \quad (20)$$

When the model parameter spaces are the same, it is common to set $c_0 = c_1$. In the case where the two models have differing dimensions, special effort has to be invested to assign a value to c_1/c_0 based on extrinsic arguments (Berger & Pericchi 2001). Therefore, imagine no counts in either region. This means no signal was observed, which means the signal hypothesis H_1 cannot become more likely

$$P(H_0|0, 0) \geq P_0(H_0). \quad (21)$$

This is a limit on the posterior model probability, which can be used as the basis of a robust Bayesian analysis (Berger et al. 1994). In particular, I argue that when no counts are observed, the probability for either model remains the same, and therefore equality holds in Equation (21). This approach leads to the determination of the fraction c_1/c_0 via the following equation:

$$\frac{c_1}{c_0} = \frac{\gamma'}{\delta'} |_{N_{\text{on}}, N_{\text{off}}=0}. \quad (22)$$

The evaluation of Equation (17) together with Equation (22) may be found in Appendix B. Altogether, the probability of H_0 being true given N_{on} and N_{off} is

$$P(H_0|N_{\text{on}}, N_{\text{off}}) = \frac{\gamma}{\gamma + c_1/c_0\delta}, \quad (23)$$

where γ and δ are defined in terms of the Gamma function $\Gamma(x)$ and the hypergeometric function ${}_2F_1(a, b; c; z)$:

$$\gamma := (1 + 2N_{\text{off}})\alpha^{1/2+N_{\text{on}}+N_{\text{off}}}\Gamma(1/2 + N_{\text{on}} + N_{\text{off}}), \quad (24)$$

$$\begin{aligned} \delta := & 2(1 + \alpha)^{N_{\text{on}}+N_{\text{off}}}\Gamma(1 + N_{\text{on}} + N_{\text{off}}) \\ & \times {}_2F_1(1/2 + N_{\text{off}}, 1 + N_{\text{on}} + N_{\text{off}}; 3/2 + N_{\text{off}}; -1/\alpha), \end{aligned} \quad (25)$$

$$\frac{c_1}{c_0} = \frac{\sqrt{\pi}}{2 \arctan(1/\sqrt{\alpha})}. \quad (26)$$

Equation (23), however, is not restricted to small count numbers and, with current PCs and numerical tools like MATHEMATICA, can be easily calculated up to thousands of counts.

3.2. Signal Detection

A signal detection based on Equation (23) may be claimed when the resulting probability of the null hypothesis H_0 is low. In high-energy astrophysics, the consensus (Li & Ma 1983; Abdo et al. 2009) p value for a source discovery is $p = 5.7 \times 10^{-7}$, corresponding to a 5σ measurement. Scientists frequently use lower thresholds for the detection of known sources. However, this value is used in this paper for the probability of H_0 as source detection criterion.

One must keep in mind that these are two completely different quantities: a probability of a model and a frequency of an outcome. $P(H_0|N_{\text{on}}, N_{\text{off}})$ explicitly weighs alternative models, while the frequentist result does not.

That said, and with the help of the inverse error function $\text{erf}^{-1}(x)$, the Bayesian significance S_b is introduced and defined as “if the probability were normally distributed, it would correspond to a S_b standard deviation measurement:”

$$S_b = \sqrt{2} \text{erf}^{-1}[1 - P(H_0|N_{\text{on}}, N_{\text{off}})]. \quad (27)$$

Using the above equation, it is easy to compare detection or discovery claims with different methods and thresholds, as shown in Section 4.

3.3. Signal Strength

If the counted events lead to a detection, then the signal parameter strength can be estimated. In other words, it is safe to assume hypothesis H_1 . The conditional probability of the signal and the background parameters λ_s and λ_{bg} may then be calculated from Bayes' law:

$$\begin{aligned} P(\lambda_s, \lambda_{\text{bg}}|N_{\text{on}}, N_{\text{off}}, H_1) \\ = \frac{P(N_{\text{on}}, N_{\text{off}}|\lambda_s, \lambda_{\text{bg}}, H_1)P_0(\lambda_s, \lambda_{\text{bg}}|H_1)}{\int_0^\infty \int_0^\infty P(N_{\text{on}}, N_{\text{off}}|\lambda_s, \lambda_{\text{bg}}, H_1)P_0(\lambda_s, \lambda_{\text{bg}}|H_1)d\lambda_s d\lambda_{\text{bg}}}. \end{aligned} \quad (28)$$

Given the data, one would like to infer the signal λ_s without reference to λ_{bg} while fully accounting for the uncertainty on λ_{bg} . This can be done by marginalizing over the nuisance parameter λ_{bg} :

$$P(\lambda_s|N_{\text{on}}, N_{\text{off}}, H_1) = \int_0^\infty P(\lambda_s, \lambda_{\text{bg}}|N_{\text{on}}, N_{\text{off}}, H_1)d\lambda_{\text{bg}}. \quad (29)$$

Equation (29) can be analytically calculated using Equations (11) and (15), and the results from Appendix B, which is done in Appendix C. The improper prior is acceptable because the proportionality constant c_1 cancels and the posterior is proper. The result may be expressed in terms of three functions, namely, the Poisson distribution $P_p(N|\lambda)$, the regularized hypergeometric function ${}_2\tilde{F}_1(a, b; c; z) = {}_2F_1(a, b; c; z)/\Gamma(c)$, and the Tricomi confluent hypergeometric function $U(a, b, z)$:

$$\begin{aligned} P(\lambda_s|N_{\text{on}}, N_{\text{off}}, H_1) = & P_p(N_{\text{on}} + N_{\text{off}}|\lambda_s) \\ & \times \frac{U[1/2 + N_{\text{off}}, 1 + N_{\text{off}} + N_{\text{on}}, (1 + 1/\alpha)\lambda_s]}{{}_2\tilde{F}_1(1/2 + N_{\text{off}}, 1 + N_{\text{off}} + N_{\text{on}}; 3/2 + N_{\text{off}}; -1/\alpha)}. \end{aligned} \quad (30)$$

This posterior contains the full information. In order to quote numbers, one may take the mode λ_s^* , which is the value of λ_s that maximizes the posterior distribution $P(\lambda_s|N_{\text{on}}, N_{\text{off}}, H_1)$, as the signal estimator. The error on the quoted signal can be evaluated from the cumulative distribution function. For instance, to get the smallest Bayesian interval (also known as highest posterior density or HPD interval) containing the signal parameter with 68% probability, one can solve

$$0.68 = \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} P(\lambda_s|N_{\text{on}}, N_{\text{off}}, H_1)d\lambda_s, \quad (31)$$

together with the constraint

$$P(\lambda_{\text{min}}|N_{\text{on}}, N_{\text{off}}, H_1) = P(\lambda_{\text{max}}|N_{\text{on}}, N_{\text{off}}, H_1), \quad (32)$$

numerically for λ_{min} and λ_{max} . The final result may be quoted as

$$\lambda_s = \lambda_{s-(\lambda_s^*-\lambda_{\text{min}})}^{*(\lambda_{\text{max}}-\lambda_s^*)}. \quad (33)$$

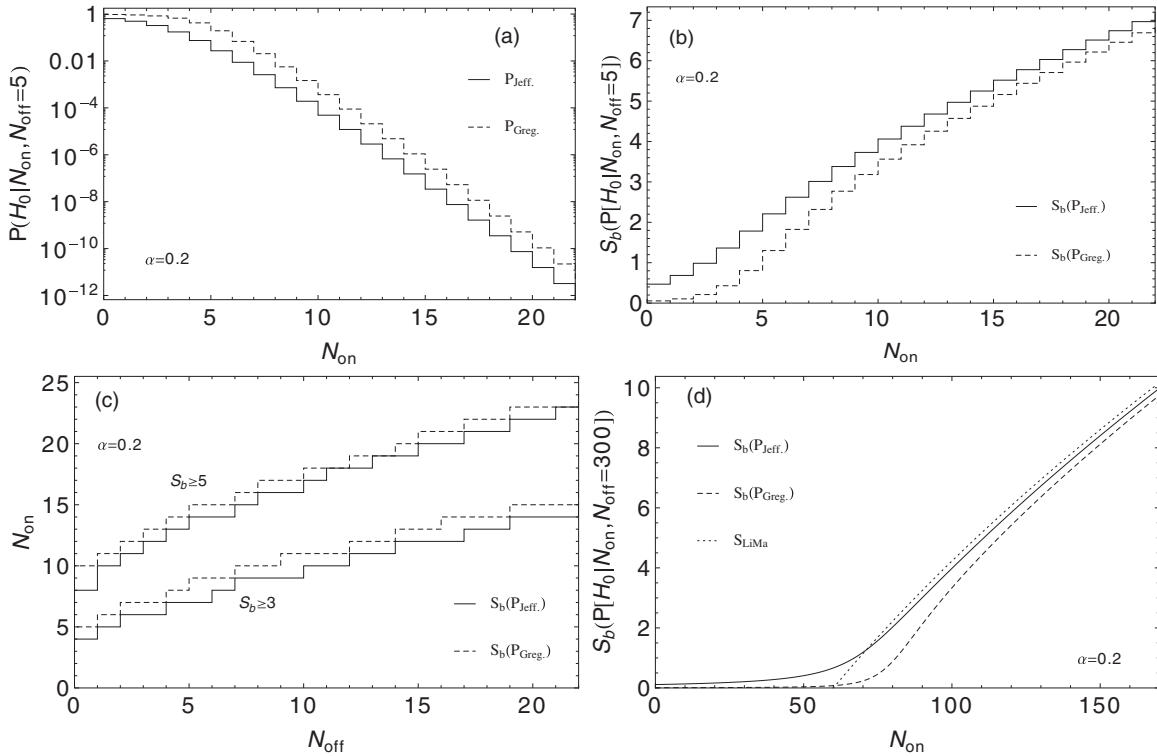


Figure 1. Comparison of the On/Off hypothesis test with Jeffreys’s and Gregory’s priors for low and large count numbers. (a) The null hypothesis posterior probability is shown as a function of N_{on} in the low counts regime. (b) Shows the same, but using the nonlinear Bayesian significance scale Equation (27). (c) The limiting curves for which $N_{\text{on}} \cdot S_b \geq 3$ or $S_b \geq 5$ are shown. (d) Shows a comparison in the large counts regime and additionally Li & Ma (1983) and Equation (17).

3.4. Signal Upper Limit

If the data show no significant detection, then an upper limit on the signal parameter may be calculated, assuming that the signal is present (i.e., H_1 is true) but too weak to be measured. For example, a 99% probability limit λ_{99} on the signal parameter λ_s is calculated by solving

$$\int_0^{\lambda_{99}} P(\lambda_s | N_{\text{on}}, N_{\text{off}}, H_1) d\lambda_s = 0.99. \quad (34)$$

This result comes naturally in a Bayesian approach of the problem but is hard to calculate in a frequentist approach. In particular, frequentists struggle with the marginalization of the problem and with special cases at the border of the parameter space, all of which lead to ad hoc adjustments without theoretical justification (Rolke et al. 2005). The only practical remedy comes from Monte Carlo studies which show that, in fact, such limits with adjustments have (at least) the claimed frequentist coverage. In this Bayesian approach, all possible values in the parameter space are dealt with in a uniform way, no matter if there are zero counts or thousands of counts. The signal upper limit result is particularly interesting for $N_{\text{on}} = N_{\text{off}} = 0$. It underlines the fact that measuring zero is different from not measuring at all, and hence valid limits can be derived. Importantly, the estimates are always physically meaningful (i.e., positive λ_s^* , λ_{min} , λ_{max} , λ_{99} , ...).

4. VALIDATION

Jeffreys’s prior is constructed by a formal rule (Jeffreys 1998) and motivated by the requirement for invariance under one-to-one transforms. However, this is not the only possible choice

and, when data are sparse, the choice of the prior is important. In order to validate that this is a reasonable choice, I compare it to the prior from Gregory (2005), the frequentist solution from Li & Ma (1983), and to a simulation.

4.1. Model Comparison

For the On/Off problem, one alternative with informative flat priors was presented by Gregory (2005). The hypothesis test, in this case, is dependent on the prior signal upper boundary λ_{smax} in addition to N_{on} , N_{off} , and α . Therefore, reasonable assumptions on the signal upper boundary λ_{smax} have to be made in order to compare Equations (14) and (24) of Gregory (2005) with Equation (23). The signal posteriors, however, can be compared directly as they depend only on the three initial parameters in both cases.

The hypothesis test comparison is shown in Figure 1. Figures 1(a)–(c) show the situation for a typical low-count case with $\alpha = 0.2$ and an assumed $\lambda_{\text{smax}} = 22$, such that a signal detection with that strength would be without any doubt. Gregory’s prior shows similar behavior to Jeffreys’s prior, but is slightly shifted towards higher probabilities for the null hypothesis $P(H_0 | N_{\text{on}}, N_{\text{off}})$ or lower significance S_b (Equation (27)). In Figure 1(c), one can see the limiting curve for which N_{on} , given N_{off} , the significance S_b is ≥ 3 or ≥ 5 . This shows that when it comes to decision making in the low-count regime, both models are mostly within one count of each other.

Figure 1(d) shows a comparison of the different priors for high-count numbers with $\alpha = 0.2$, $N_{\text{off}} = 300$, and $\lambda_{\text{smax}} = 170$. Additionally, the methods are compared to the frequentist result of Li & Ma (1983) and Equation (17). Both methods appear to converge on Li & Ma’s result for large number counts, but Jeffreys’s prior gives a closer approximation. From

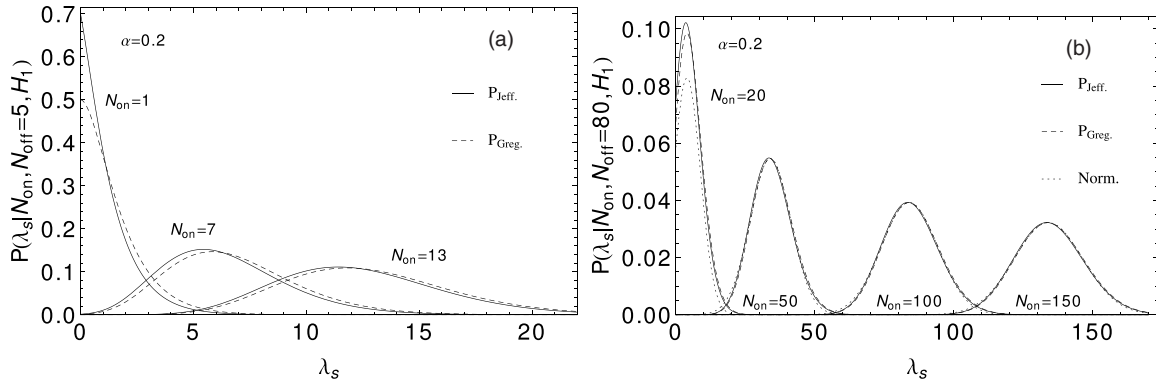


Figure 2. Comparison of the signal posteriors with Jeffreys's and Gregory's priors. (a) A comparison for low count numbers. (b) A comparison for high count numbers and a normal distribution with mean Equation (35) and variance Equation (36)

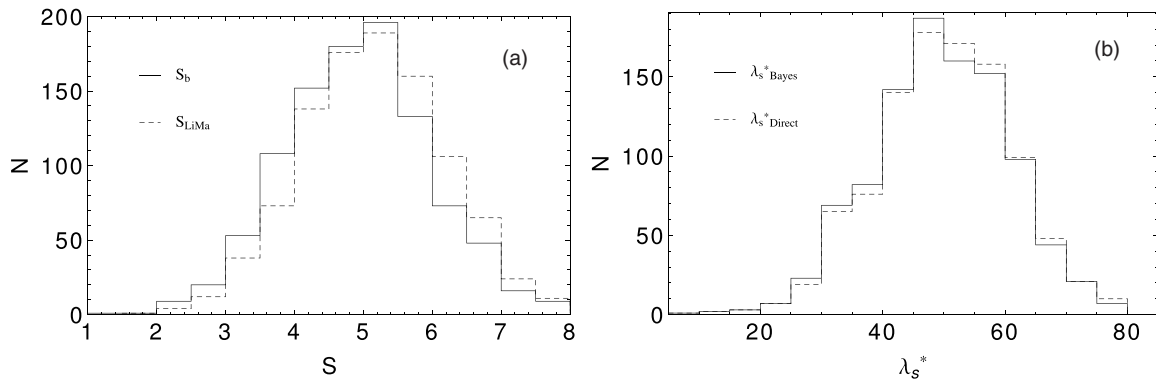


Figure 3. Simulation results of the analysis compared to Li & Ma (1983) and direct background subtraction.

a physical point of view, one could argue that for $N_{\text{on}} \sim \alpha N_{\text{off}}$, $P(H_0 | N_{\text{on}}, N_{\text{off}})$ should be close to the prior model probability $P_0(H_0) = 0.5$ because any one N_{on} more should increase the probability for the signal hypothesis H_1 and any N_{on} less should decrease it. Numerically, it seems that this is the case for Jeffrey's prior but not for Gregory's prior.

When comparing the signal posterior, the priors of Jeffrey and Gregory agree well. Figure 2(a) shows a comparison of the two methods for low count numbers with $\alpha = 0.2$ and $N_{\text{off}} = 5$. The differences in the signal posterior are marginal. For high count numbers, as shown in Figure 2(b) with $\alpha = 0.2$ and $N_{\text{off}} = 80$, both methods converge quickly to the classical result of a normal distribution with

$$E(\lambda_s) = N_{\text{on}} - \alpha N_{\text{off}} \quad (35)$$

$$\text{Var}(\lambda_s) = N_{\text{on}} + \alpha^2 N_{\text{off}}. \quad (36)$$

However, due to subtraction and when $N_{\text{on}} \sim \alpha N_{\text{off}}$, the normal distribution can include negative values for the signal confidence region. This problem is resolved using the Bayesian methods.

Overall, the results are comparable and show that the results obtained by Jeffrey's prior are sensible. They behave well in all test-case examples, in particular, at $N_{\text{on}} \sim \alpha N_{\text{off}}$, and converge to the other results for high count numbers.

4.2. Simulations

To further verify the method developed in this paper, the hypothesis test and the maximum signal posterior are calculated

for a simulated set of observations. First, 1000 N_{on} and N_{off} are drawn randomly from two Poisson distributions with the means of Equations (7) and (9). Second, the developed methods are applied to the simulation, as is Li & Ma's method for the hypothesis test and a direct background subtraction (Equation (35)) for the signal strength. Then, the results of these methods are compared to one another and to the true parameters. These are $\lambda_{\text{bg}} = 300$ for the background strength and $\lambda_s = 50$ for the signal strength. The ratio of exposure α is 1/5.

Figure 3(a) shows the hypothesis test simulation results. Li & Ma's test statistic shows a small systematic shift compared to the Bayesian significance (Equation (27)) in agreement with the results from Section 4.1. In Figure 3(b), the signal strength comparison is shown. The parameter λ_s^* is first calculated as the maximum of the marginalized posterior (Equation (30)) and then with direct background subtraction. Both methods agree well and can reconstruct the true signal parameter $\lambda_s = 50$ with similar errors.

5. APPLICATION: GAMMA-RAY BURSTS

Gamma-ray bursts (GRB) are extraterrestrial flashes of gamma-rays mostly lasting only a few seconds. One interesting question is whether GRBs produce very high-energy (>100 GeV) gamma-rays, as proposed by some theories (e.g., Abdo et al. 2009). Because of the flares' durations and fluences, gamma-ray satellites and Cherenkov telescopes measure only a few events during the flare itself or shortly thereafter. In Table 1, data from 12 GRB observations observed by the

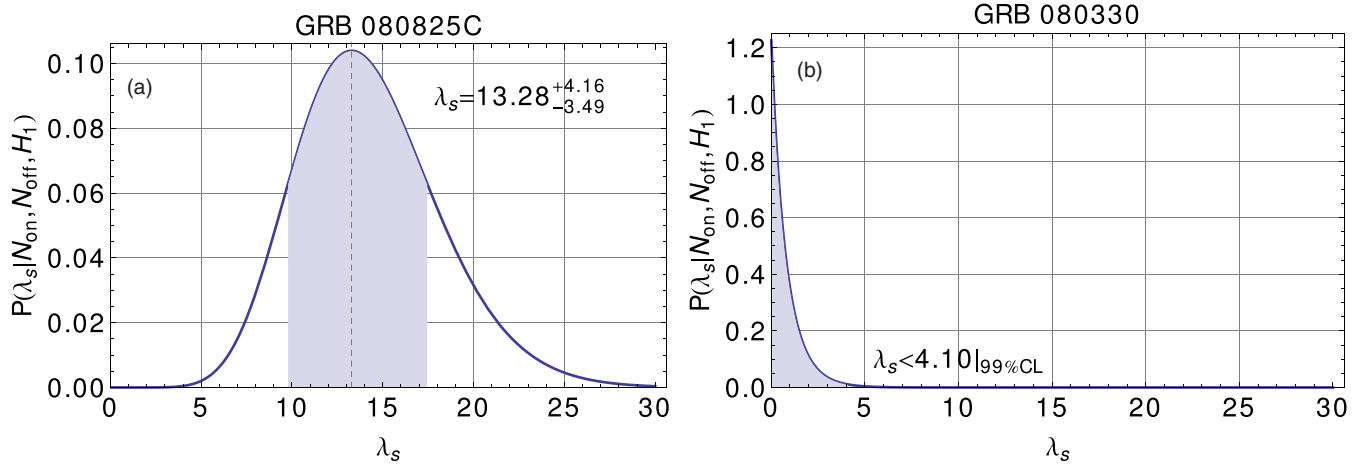


Figure 4. Comparison of the marginal signal parameter λ_s posterior distribution for two gamma-ray bursts. In (a), GRB 080825C, a source is detected and the signal parameter estimate and its smallest 68% credibility interval is calculated. In (b), an upper limit is calculated for GRB 080330 where no on event was measured. (A color version of this figure is available in the online journal.)

Table 1
Low Count Gamma-ray Burst Data

GRB	N_{on}	N_{off}	α	S_{lm}	$P(H_0 N_{\text{on}}, N_{\text{off}})$	S_b	λ_s	λ_{99}	$\lambda_{\text{Ref.}}$	$\lambda_{99}^{\text{Rolke}}$	Reference
070419A	2	14	0.057	...	0.28	1.09	...	6.88	...	7.34	1
070521	3	113	0.057	...	0.72	0.36	...	6.12	...	3.52	1
070612B	3	21	0.066	...	0.26	1.12	...	8.00	...	8.54	1
080310	3	23	0.128	...	0.51	0.66	...	7.16	...	7.08	1
080330	0	15	0.123	...	0.70	0.39	...	4.10	...	2.40	1
080604	2	40	0.063	...	0.58	0.55	...	6.12	...	5.66	1
080607	4	16	0.112	...	0.21	1.27	...	9.17	...	9.83	1
080825C	15	19	0.063	6.4	9.66E-10	6.11	$13.28^{+4.16}_{-3.49}$...	13.7	...	2
081024A	1	7	0.142	...	0.50	0.67	...	5.29	...	5.19	1
090418A	3	16	0.123	...	0.37	0.89	...	7.64	...	8.01	1
090429B	2	7	0.106	...	0.26	1.13	...	6.92	...	7.41	1
090515	4	24	0.126	...	0.41	0.83	...	8.34	...	8.66	1

Notes. Table 1 shows low-count gamma-ray burst data where either or both of N_{on} or N_{off} are ≤ 15 . Only GRB 080825C was detected with a high significance. Its reference source strength $\lambda_{\text{Ref.}}$ is given in Abdo et al. (2009). All calculated values are in bold. The ratio of exposure α is calculated according to Acciari et al. (2011). The α values are then used for calculating Rolke's upper limit $\lambda_{99}^{\text{Rolke}}$.

References. (1) Acciari et al. 2011; (2) Abdo et al. 2009.

space-based *Fermi* Large Area Telescope (*Fermi*-LAT) and the ground-based VERITAS Cherenkov telescope are compiled.¹

Due to the difficulty of the detection, those events usually report an upper limit with low statistics. Only for the *Fermi*-LAT GRB 080825C was there significant evidence to report a discovery. For this GRB, the probability of the background-only model $P(H_0|N_{\text{on}}, N_{\text{off}})$ is 9.66×10^{-10} and the GRB is therefore detected. The significance expressed on the nonlinear scale (Equation (27)) is $S_b = 6.11$. This is comparable to the Li & Ma result of $S_{\text{lm}} = 6.4$.

In the second step, the most likely value of the signal parameter and the smallest 68% credibility interval are calculated. The method is demonstrated in Figure 4. The result is that $\lambda_s = 13.28^{+4.16}_{-3.49}$, which is in good agreement with the published reference of $\lambda_{\text{Ref.}} = 13.7$.

¹ The α parameter of the VERITAS observations is calculated with their employed test statistic *ratio of Poisson means*, according to Cousins et al. (2008). It is solved numerically for α , except for the case of GRB 080330 where not a single count was measured in the *on* region and no significance is given. In this case, the α parameter is assumed to be the mean of the other VERITAS wobble-mode observations.

In the same figure, an observation from GRB 080330, for which discovery cannot be claimed, is shown. For GRB 080330 and all other GRBs, the data do not show evidence for a gamma-ray source. In this case, upper limits λ_{99} are calculated. All results are summarized in Table 1 and are compared to the Rolke 99% upper limits $\lambda_{99}^{\text{Rolke}}$, which VERITAS used. GRB 080330 is special in the sense that not even one *on* event was measured. The results are mostly in good agreement but, especially at the border of the parameter space for $N_{\text{on}} \leq \alpha N_{\text{off}}$, also deviate from another and show the limit of Rolke's method, which are overcome by the Bayesian method.

6. CONCLUSION

Many particle physicists, cosmic-ray physicists, and high-energy astrophysicists struggle with sparse On/Off data. With this new Bayesian method which overcomes the weaknesses of the currently used methods, it is possible to go down to single count On/Off measurements. Claiming detections, setting credibility intervals, or setting upper limits is unified in a single and consistent method.

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APPENDIX A JEFFREYS'S PRIORS

Calculation of the prior of λ_{bg} in the H_0 model:

$$P_0(\lambda_{\text{bg}}|H_0) = \left(- \sum_{N_{\text{on}}=0}^{\infty} \sum_{N_{\text{off}}=0}^{\infty} \times \left\{ \partial_{\lambda_{\text{bg}}}^2 \ln [P_{\text{P}}(N_{\text{on}}|\alpha\lambda_{\text{bg}})P_{\text{P}}(N_{\text{off}}|\lambda_{\text{bg}})] \right\} \times P_{\text{P}}(N_{\text{on}}|\alpha\lambda_{\text{bg}})P_{\text{P}}(N_{\text{off}}|\lambda_{\text{bg}}) \right)^{\frac{1}{2}} = \sqrt{\frac{1+\alpha}{\lambda_{\text{bg}}}}. \quad (\text{A1})$$

Calculation of the prior for λ_s and λ_{bg} in the H_1 model:

$$I_{s,s} = - \sum_{N_{\text{on}}=0}^{\infty} \sum_{N_{\text{off}}=0}^{\infty} \left\{ \partial_{\lambda_s}^2 \ln [P_{\text{P}}(N_{\text{on}}|\lambda_s + \alpha\lambda_{\text{bg}})P_{\text{P}}(N_{\text{off}}|\lambda_{\text{bg}})] \right\} \times P_{\text{P}}(N_{\text{on}}|\lambda_s + \alpha\lambda_{\text{bg}})P_{\text{P}}(N_{\text{off}}|\lambda_{\text{bg}}) = \frac{1}{\alpha\lambda_{\text{bg}} + \lambda_s}, \quad (\text{A2})$$

$$I_{\text{bg},\text{bg}} = - \sum_{N_{\text{on}}=0}^{\infty} \sum_{N_{\text{off}}=0}^{\infty} \left\{ \partial_{\lambda_{\text{bg}}}^2 \ln [P_{\text{P}}(N_{\text{on}}|\lambda_s + \alpha\lambda_{\text{bg}})P_{\text{P}}(N_{\text{off}}|\lambda_{\text{bg}})] \right\} \times P_{\text{P}}(N_{\text{on}}|\lambda_s + \alpha\lambda_{\text{bg}})P_{\text{P}}(N_{\text{off}}|\lambda_{\text{bg}}) = \frac{\alpha\lambda_{\text{bg}} + \alpha^2\lambda_{\text{bg}} + \lambda_s}{\lambda_{\text{bg}}(\alpha\lambda_{\text{bg}} + \lambda_s)}, \quad (\text{A3})$$

$$I_{s,\text{bg}} = I_{\text{bg},s} = - \sum_{N_{\text{on}}=0}^{\infty} \sum_{N_{\text{off}}=0}^{\infty} \left\{ \partial_{\lambda_{\text{bg}}} \partial_{\lambda_s} \ln [P_{\text{P}}(N_{\text{on}}|\lambda_s + \alpha\lambda_{\text{bg}})P_{\text{P}}(N_{\text{off}}|\lambda_{\text{bg}})] \right\} \times P_{\text{P}}(N_{\text{on}}|\lambda_s + \alpha\lambda_{\text{bg}})P_{\text{P}}(N_{\text{off}}|\lambda_{\text{bg}}) = \frac{\alpha}{\alpha\lambda_{\text{bg}} + \lambda_s}. \quad (\text{A4})$$

The off-diagonal elements are equal because the matrix is symmetric (symmetry of second derivatives). The final result for $P_0(\lambda_s, \lambda_{\text{bg}}|H_1)$ is

$$P_0(\lambda_s, \lambda_{\text{bg}}|H_1) = \left\{ \det [I(\lambda_s, \lambda_{\text{bg}}|H_1)] \right\}^{\frac{1}{2}} = \sqrt{\frac{1}{\lambda_{\text{bg}}(\alpha\lambda_{\text{bg}} + \lambda_s)}}. \quad (\text{A5})$$

APPENDIX B

CALCULATION OF THE PROBABILITY OF H_0

For the calculation of Equation (17), one must solve three parts. First,

$$\int_0^{\infty} P(N_{\text{on}}, N_{\text{off}}|\lambda_{\text{bg}}, H_0)P_0(\lambda_{\text{bg}}|H_0)d\lambda_{\text{bg}} = \int_0^{\infty} \frac{e^{-\lambda_{\text{bg}}(1+\alpha)}\lambda_{\text{bg}}^{N_{\text{off}}}(\alpha\lambda_{\text{bg}})^{N_{\text{on}}}}{N_{\text{on}}!N_{\text{off}}!} \sqrt{\frac{1+\alpha}{\lambda_{\text{bg}}}} d\lambda_{\text{bg}} = \frac{\alpha^{N_{\text{on}}}(1+\alpha)^{-N_{\text{on}}-N_{\text{off}}}\Gamma\left(\frac{1}{2} + N_{\text{on}} + N_{\text{off}}\right)}{N_{\text{off}}!N_{\text{on}}!}, \quad (\text{B1})$$

and second,

$$\int_0^{\infty} \int_0^{\infty} P(N_{\text{on}}, N_{\text{off}}|\lambda_s, \lambda_{\text{bg}}, H_1)P_0(\lambda_s, \lambda_{\text{bg}}|H_1)d\lambda_s d\lambda_{\text{bg}} = \int_0^{\infty} P_{\text{P}}(N_{\text{off}}|\lambda_{\text{bg}}) \sqrt{\frac{1}{\lambda_{\text{bg}}}} \times \int_0^{\infty} P_{\text{P}}(N_{\text{on}}|\lambda_s + \alpha\lambda_{\text{bg}}) \sqrt{\frac{1}{\lambda_s + \alpha\lambda_{\text{bg}}}} d\lambda_s d\lambda_{\text{bg}} = \int_0^{\infty} P_{\text{P}}(N_{\text{off}}|\lambda_{\text{bg}}) \sqrt{\frac{1}{\lambda_{\text{bg}}}} \frac{\Gamma\left(\frac{1}{2} + N_{\text{on}}, \alpha\lambda_{\text{bg}}\right)}{N_{\text{on}}!} d\lambda_{\text{bg}}. \quad (\text{B2})$$

$\Gamma(a, z)$ stands for the upper incomplete gamma function. The remaining integral with respect to λ_{bg} yields

$$= \frac{1}{N_{\text{on}}!N_{\text{off}}!} \frac{2\alpha^{-\frac{1}{2}-N_{\text{off}}}}{1+2N_{\text{off}}}\Gamma(1+N_{\text{on}}+N_{\text{off}}) \times {}_2F_1\left(\frac{1}{2} + N_{\text{off}}, 1 + N_{\text{on}} + N_{\text{off}}; \frac{3}{2} + N_{\text{off}}; -\frac{1}{\alpha}\right). \quad (\text{B3})$$

By inserting Equations (B1) and (B3) into Equation (17) and simplifying, one finds the solution

$$P(H_0|N_{\text{on}}, N_{\text{off}}) = \frac{\gamma}{\gamma + c_1/c_0\delta}, \quad (\text{B4})$$

$$\gamma := (1 + 2N_{\text{off}})\alpha^{1/2+N_{\text{on}}+N_{\text{off}}} \times \Gamma(1/2 + N_{\text{on}} + N_{\text{off}}), \quad (\text{B5})$$

$$\delta := 2(1 + \alpha)^{N_{\text{on}}+N_{\text{off}}}\Gamma(1 + N_{\text{on}} + N_{\text{off}}) \times {}_2F_1(1/2 + N_{\text{off}}, 1 + N_{\text{on}} + N_{\text{off}}; 3/2 + N_{\text{off}}; -1/\alpha). \quad (\text{B6})$$

The constant fraction c_1/c_0 is calculated with Equation (22), inserting the above results. Equation (23) follows.

APPENDIX C

CALCULATION OF THE MARGINALIZED POSTERIOR FOR THE SIGNAL PARAMETER

The denominator of Equation (28) is given in Equation (B3) and does not depend on the parameter λ . The problem is therefore reduced to calculating the integral:

$$\int_0^{\infty} P(N_{\text{on}}, N_{\text{off}}|\lambda_s, \lambda_{\text{bg}}, H_1)P_0(\lambda_s, \lambda_{\text{bg}}|H_1)d\lambda_{\text{bg}} = \int_0^{\infty} P_{\text{P}}(N_{\text{off}}|\lambda_{\text{bg}})P_{\text{P}}(N_{\text{on}}|\lambda_s + \alpha\lambda_{\text{bg}}) \times \sqrt{\frac{1}{\lambda_{\text{bg}}(\lambda_s + \alpha\lambda_{\text{bg}})}} d\lambda_{\text{bg}} = \frac{e^{-\lambda_s}}{N_{\text{on}}!N_{\text{off}}!} \times \int_0^{\infty} (\lambda_s + \alpha\lambda_{\text{bg}})^{N_{\text{on}}-\frac{1}{2}}\lambda_{\text{bg}}^{N_{\text{off}}-\frac{1}{2}}e^{-\lambda_{\text{bg}}(1+\alpha)} d\lambda_{\text{bg}}, \quad (\text{C1})$$

which resembles the integral representation of the Tricomi confluent hypergeometric function (see, for instance, NIST 2013,

Equation (13.4.4)). By substituting the integration variable λ_{bg} with

$$\lambda = \frac{\alpha \lambda_{\text{bg}}}{\lambda_s}, \quad (\text{C2})$$

one finds the result

$$\frac{e^{-\lambda_s}}{N_{\text{on}}! N_{\text{off}}!} \lambda_s^{N_{\text{on}} - \frac{1}{2}} \left(\frac{\lambda_s}{\alpha} \right)^{N_{\text{off}} + \frac{1}{2}} \Gamma\left(\frac{1}{2} + N_{\text{off}}\right) \times U\left[\frac{1}{2} + N_{\text{off}}, 1 + N_{\text{on}} + N_{\text{off}}, \left(1 + \frac{1}{\alpha}\right) \lambda_s\right]. \quad (\text{C3})$$

Equations (C3) and (B3), combined in Equation (28), simplified with Equation (1) and the regularized hypergeometric function ${}_2\tilde{F}_1(a, b; c; z) = {}_2F_1(a, b; c; z)/\Gamma(c)$, give the final result for the marginalized posterior for λ_s :

$$P(\lambda_s | N_{\text{on}}, N_{\text{off}}, H_1) = P_{\text{P}}(N_{\text{on}} + N_{\text{off}} | \lambda_s) \times \frac{U[1/2 + N_{\text{off}}, 1 + N_{\text{off}} + N_{\text{on}}, (1 + 1/\alpha)\lambda_s]}{{}_2\tilde{F}_1(1/2 + N_{\text{off}}, 1 + N_{\text{off}} + N_{\text{on}}; 3/2 + N_{\text{off}}; -1/\alpha)}.$$

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