## 5. Misure di capacità termica a bassa temperatura

Si trova sperimentalmente che la capacità termica molare (a volume costante) dei solidi elementari ha un valore approssimativamente costante ad alta temperatura (legge di Dulong e Petit), mentre diminuisce – in misura diversa per diverse sostanze – al diminuire della temperatura. La figura seguente mostra ad esempio il calore specifico del rame in funzione della temperatura.
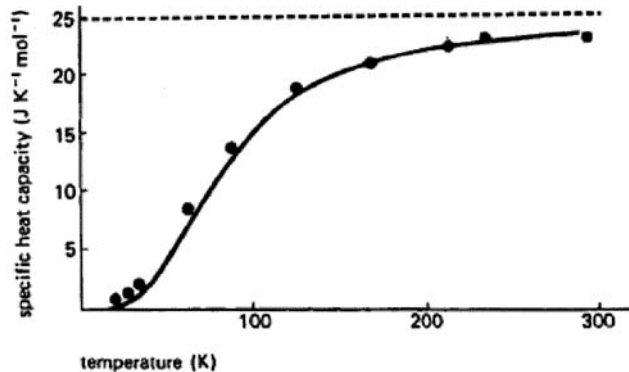
Valori sperimentali della capacità termica del rame (la curva mostra l'andamento ottenuto dalla teoria elementare di Einstein (1907)).

C'è un modo estremamente semplice per dedurre il valore asintotico che si osserva ad alta temperatura. Consideriamo un reticolo cristallino di *N* atomi: questi atomi oscillano intorno alle loro posizioni di equilibrio nel reticolo cristallino, e ciascuno di essi ha 3 gradi di libertà. Ad ogni grado di libertà è associata sia un'energia cinetica, sia un'energia potenziale, e poichè il valore medio di ciascuna componente quadratica dell'energia contribuisce con $kT/2$ all'energia media totale (principio di equipartizione dell'energia), l'energia media totale è $U = 3NkT = 3nRT$. Poiché $C = \partial U/\partial T$, allora la capacità termica molare è $3R \approx 24.94$ J/mole·K.

La meccanica quantistica porta ad una modifica del principio di equipartizione dell'energia: Einstein è stato il primo a produrre un'approssimazione che tiene conto della meccanica quantistica. L'approssimazione di Einstein funziona male a temperature molto basse ma è sufficiente per descrivere l'andamento globale della capacità termica molare (v. figura).

La verifica della formula di Einstein richiede misure fatte a diverse temperature: in questo esperimento ci limitiamo ad una sola temperatura, quella di ebollizione dell'azoto liquido ($\approx$ 77 K; questo ci permette di fare un semplice confronto con la capacità termica a temperatura ambiente).

Lo schema di esperimento è quello riportato nell'articolo di Thompson e White, riprodotto alla fine di questa nota.

Appendice: il modello di Einstein

Come funziona l'approssimazione di Einstein? Per capirlo dobbiamo assumere come noto un risultato della Meccanica Quantistica, vale a dire che un oscillatore ha livelli energetici

$$\varepsilon_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

dove ω è la frequenza naturale dell'oscillatore (se κ è la costante elastica e μ la massa, allora $\omega = \sqrt{\kappa/\mu}$). Perciò l'energia media di un oscillatore quantistico è data da

$$
\begin{aligned}
\overline{\varepsilon} &= \frac{\sum_{n=0}^{\infty} \varepsilon_n \exp\left(-\frac{\varepsilon_n}{kT}\right)}{\sum_{n=0}^{\infty} \exp\left(-\frac{\varepsilon_n}{kT}\right)} = \frac{\sum_{n=0}^{\infty} \varepsilon_n \exp(-\beta\varepsilon_n)}{\sum_{n=0}^{\infty} \exp(-\beta\varepsilon_n)} = -\frac{\partial}{\partial\beta} \ln \sum_{n=0}^{\infty} \exp(-\beta\varepsilon_n) \\
&= -\frac{\partial}{\partial\beta} \ln \sum_{n=0}^{\infty} \exp\left[-\beta\hbar\omega\left(n+\frac{1}{2}\right)\right] = -\frac{\partial}{\partial\beta} \ln\left\{ e^{-\frac{\beta\hbar\omega}{2}} \sum_{n=0}^{\infty} e^{-n\beta\hbar\omega}\right\} \\
&= -\frac{\partial}{\partial\beta} \ln \frac{e^{-\frac{\beta\hbar\omega}{2}}}{1 - e^{-\beta\hbar\omega}} = -\frac{\partial}{\partial\beta}\left[-\frac{\beta\hbar\omega}{2} - \ln\left(1 - e^{-\beta\hbar\omega}\right)\right] \\
&= \frac{\hbar\omega}{2} + \hbar\omega \frac{e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} = \hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1}\right) \\
&= k\left(\frac{\hbar\omega}{k}\right)\left(\frac{1}{2} + \frac{1}{e^{(\hbar\omega/k)/T} - 1}\right) = k\theta_E\left(\frac{1}{2} + \frac{1}{e^{\theta_E/T} - 1}\right)
\end{aligned}
$$

dove $\theta_E = \dfrac{\hbar\omega}{k}$ è la temperatura di Einstein.

Se si assume che gli atomi in un cristallo si possano approssimare come oscillatori semplici, ciascuno con 3 gradi di libertà, allora in totale ci sono $3N$ gradi di libertà e l'energia media è

$$U = 3Nk\theta_E\left(\frac{1}{2} + \frac{1}{e^{\theta_E/T} - 1}\right)$$

Si noti che se il rapporto θ_E/T è molto piccolo, allora

$$U \approx 3Nk\theta_E\left(\frac{1}{2} + \frac{T}{\theta_E}\right) \approx 3NkT$$

e si ritrova il risultato che ci aveva portato alla legge di Dulong e Petit.

Dalla formula per l'energia interna media, e assumendo $N = N_A$, si trova la capacità termica molare

$$\frac{\partial}{\partial T}\left[3R\theta_E\left(\frac{1}{2} + \frac{1}{e^{\theta_E/T} - 1}\right)\right] = 3R\frac{\theta_E^2}{T^2}\frac{e^{\theta_E/T}}{\left(e^{\theta_E/T} - 1\right)^2}$$

Questa capacità termica dipende solo dal rapporto $\dfrac{\theta_E}{T} = \dfrac{\hbar\omega}{kT} = \dfrac{\hbar\sqrt{\kappa/\mu}}{kT}$, e che se il rapporto è molto piccolo, allora la capacità termica vale approssimativamente

$$3R\frac{\theta_E^2}{T^2}\frac{1}{\left(\theta_E/T\right)^2} = 3R$$

e in questo caso si riottiene la legge di Dulong e Petit.

Se invece il rapporto è grande, allora

$$3R\frac{\theta_E^2}{T^2}\frac{e^{\theta_E/T}}{\left(e^{\theta_E/T} - 1\right)^2} \approx 3R\frac{\theta_E^2}{T^2}e^{-\theta_E/T}$$

e questa espressione tende a zero (quindi la legge di Dulong e Petit è violata).

La temperatura di Einstein è grande per solidi con grande costante elastica (molto rigidi) oppure con massa piccola (nuclei leggeri): per esempio il diamante ha una temperatura di Einstein di circa 1450 K e la sua capacità termica si avvicina a quella della legge di Dulong e Petit solo ad alta temperatura.

British ball can be readily determined:

$$\frac{c_2}{m} = \left(\frac{C_D}{m}\right)A \tfrac{1}{2}\rho, \quad A = 0.0143 \text{ ft}^2,$$

$$\rho = 0.002\,37 \text{ slugs/ft}^3, \quad m = (1.62/16)/32 \text{ slugs},$$

$$C_D = 0.36.$$

[11]Reference 5(b), p. 161.
[12]Reference 5(b), p. 160.
[13]Reference 5(b), p. 164.
[14]Reference 5(b), pp. 164–165.
[15]Reference 5(b), p. 165.

Latent heat and low-temperature heat capacity experiment for the general physics laboratory

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An experiment to measure the latent heat of vaporization of liquid nitrogen and the average heat capacities at constant pressure of several materials in the temperature range 77–295 K is described. This exercise uses normally available laboratory apparatus and is suitable for the general physics laboratory.

I. INTRODUCTION

We have recently developed a laboratory experiment for our general physics courses that incorporates the use of a cryogenic fluid (liquid nitrogen—hereafter called LN$_2$), the measurement of its latent heat of vaporization, and the study of the (average) low-temperature heat capacities of several materials and their relation to the law of Dulong and Petit. The experiment is simple and uses normally available laboratory apparatus (assuming a source of LN$_2$ is available). Data collection and analysis can be performed in a two-hour laboratory period. The values obtained have small experimental uncertainties and agree well with values calculated from published data.

II. LATENT HEAT OF VAPORIZATION OF LIQUID NITROGEN

A schematic of the apparatus used to measure the latent heat of vaporization of LN$_2$ is shown in Fig. 1. An electrical heater R was suspended in a LN$_2$ bath in a cup assembly which rested on a scale balance pan. As the LN$_2$ boiled away a background loss rate was determined by measuring $m(t)$, the mass of the LN$_2$ plus cup assembly, as a function of time. After a few minutes of observation, switch S was closed. The current I and voltage V were measured. Simultaneously, measurements of $m(t)$ continued but the rate was, of course, much faster. A good technique to measure $m(t)$ is to unbalance the scale one or two grams too light and read the time as the pointer passes zero, then unbalance again and repeat. A suitably damped dial-type balance makes this process rapid and simple.

After a few grams of LN$_2$ had been boiled off by the heater, switch S was opened again and the heater time interval, Δt, carefully noted. (A separate timer for this purpose is advantageous). Measurements of $m(t)$ were continued for a few minutes to re-establish the background. These data were plotted as shown in Fig. 2.

The double-wall styrofoam cup assembly, labeled item $\mathscr{C}$ in Fig. 1, was made by suspending a 6-oz cup inside a 14-oz cup. A styrofoam ring was used to center the cups and a bead of silicone bathtup-type sealant sealed the joint and stabilized the assembly. The analytical balance[1] had a 0–10 g dial. The heater probe had a 33-Ω, 10-W wire wound resistor (nominal value) mounted on the end of a small stainless steel tube. The power source was a 28-V (nominal) laboratory supply. The 10-W rating of $\mathscr{R}$ was exceeded but the LN$_2$ cooling compensated. The current and voltage were measured using Keithley Model 130 digital multimeters.[2] Values obtained for the data shown in Fig. 2 were $V = 28.5$ V and $I = 0.871$ A.

Straight lines were drawn through the data of Fig. 2. From the vertical displacement of the two lines near the center of the heating period the mass of the LN$_2$ boiled off was determined to be 9.35 g. The heating interval was timed to be 73.4 s. The latent heat of vaporization L_v was calculated from the relation

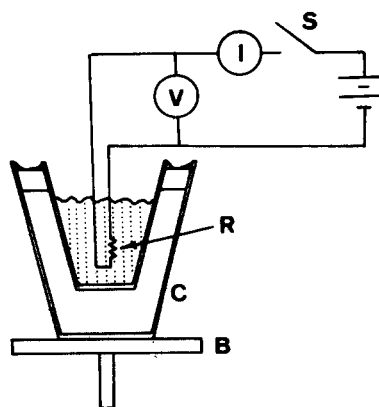$$L_v = VI\Delta t/\Delta M. \tag{1}$$

Fig. 1. Schematic of the apparatus used to measure the latent heat of vaporization of LN$_2$. The letters refer to the following items: B, balance; C, styrofoam cup assembly; I, current meter; R, 33 Ω resistor; S, switch; V, voltmeter. The power source was 28 V.

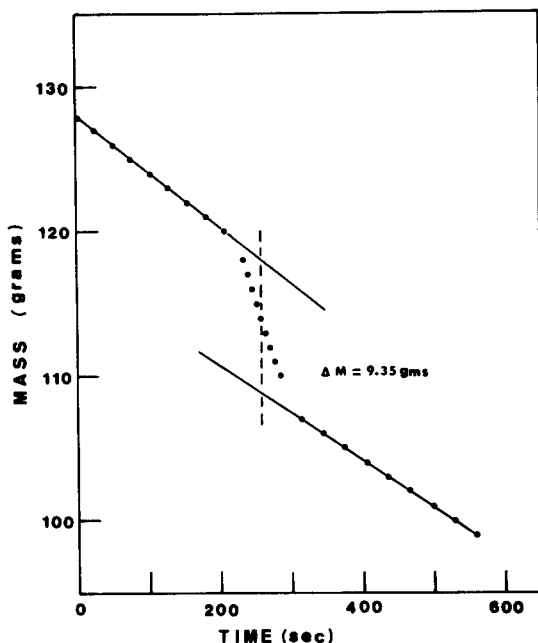© 1983 American Association of Physics Teachers

Fig. 2. Plot of the data which shows the mass of the cup assembly plus LN_2 as a function of time for the heat of vaporization of liquid nitrogen measurement. The background rates have been extrapolated to the center of the heating period which is marked by a dashed line.

The above data gives

$$L_v = \frac{28.5 \times 0.871 \times 73.4}{9.35} = 195 \text{ J/g}. \tag{2}$$

The accepted value is 197.7 J/g, or 47.23 cal/g.

III. SPECIFIC HEATS OF LEAD AND BERYLLIUM

Our students measure the molar specific heat capacities, averaged over the temperature range of 77–295 K, for two metals—lead and beryllium. These two metals were chosen since their molar heat capacities at room temperature are quite different. Lead has a value near $3R$ whereas beryllium is much lower. R is the universal gas constant. If beryllium should be hard to obtain (its toxicity makes cutting pieces to size totally out of the question in most machine shops) or too costly, silicon or carbon would be acceptable substitutes.

The heat capacity measurements were made as follows. The styrofoam cup was filled with LN_2 and, together with the metal piece to be measured, placed on the balance pan as illustrated in Fig. 3. As before, the mass $m(t)$ was measured for a few minutes to establish a background loss rate. The metal was then moved from the scale pan and gently placed into the cup of LN_2 using tongs. Measurements of $m(t)$ continued until the background rate was re-established. Extrapolation of the background rates permitted determination of the mass of liquid which boiled off in cooling the piece of metal from room temperature to 77 K. The average specific heat was then calculated from the expression

$$\overline{C}_p = \frac{(198)\Delta m}{n(T_0 - 77)} \text{ J/mole K}, \tag{3}$$

where Δm is the mass of LN_2 boiled off in units of grams, n is the amount of sample in moles, and T_0 is the original (room) temperature of the metal in degrees Kelvin.
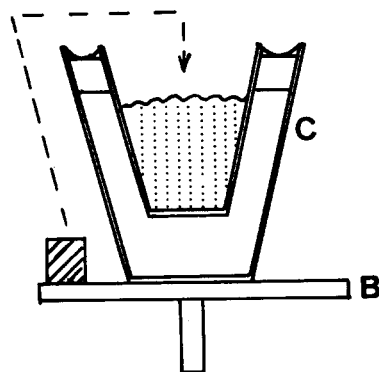

Fig. 3. Schematic diagram illustrating that the metal sample (lead or beryllium) is tranferred from the balance pan into the liquid nitrogen during the heat capacity measurements. The transfer is made after about 200 s, as shown by the data in Fig. 4. B is the balance pan and C is the cup assembly.

Figure 4 shows data obtained for lead and beryllium samples of masses 57.0 g (0.275 mole) and 3.62 g (0.402 mole), respectively. Based on this data the average specific heats at constant pressure $\overline{C}_p$ were calculated to be

$$\overline{C}_p(\text{lead}) = \frac{(198)(7.60)}{(0.275)(295 - 77)} = 25.1 \text{ J/mole K} \tag{4}$$

$$= 6.00 \text{ cal/mole K} \tag{5}$$

and

$$\overline{C}_p(\text{beryllium}) = \frac{(198)(4.30)}{(0.402)(295 - 77)} = 9.72 \text{ J/mole K} \tag{6}$$

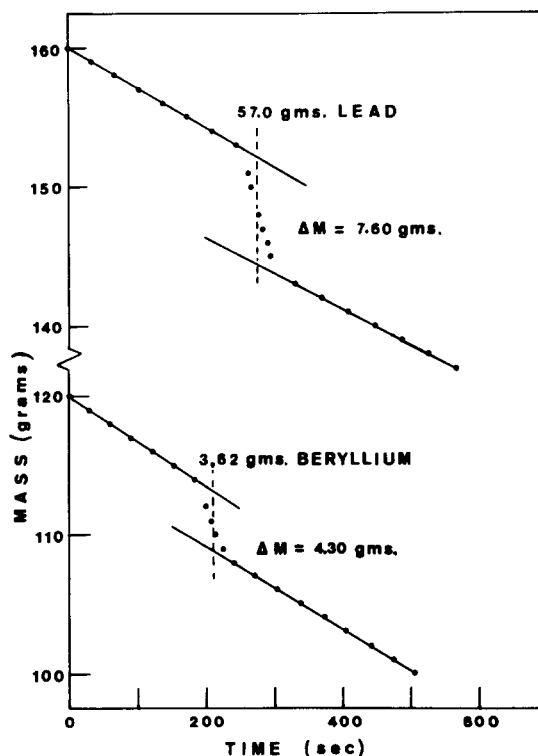$$= 2.32 \text{ cal/mole K}. \tag{7}$$

Fig. 4. Plots of the data which show the mass of the sample cup assembly plus LN_2 as a function of time for the heat capacity measurements for lead (upper) and beryllium (lower). The background rates have been extrapolated into the region where the cooling of the sample occurred. The mass of the nitrogen boil off Δm and the mass of each sample are shown for the two measurements.

Table I. Molar specific heats at constant pressure at several temperatures for Be, C(graphite), Si, Cu, and Pb. The units are cal/mole K and degrees Kelvin.[a]

T	Be	C(graphite)	Si	Cu	Pb
70	0.121	0.221	1.02	2.63	5.54
80	0.195	0.278	1.24	3.11	5.64
90	0.299	0.339	1.48	3.52	5.74
100	0.428	0.402	1.72	3.86	5.84
120	0.743	0.539	2.17	4.37	5.94
140	1.130	0.689	2.59	4.75	5.99
160	1.557	0.849	2.99	5.04	6.09
180	1.983	1.019	3.36	5.25	6.14
200	2.39	1.188	3.68	5.40	6.19
220	2.78	1.36	3.95	5.53	6.24
240	3.17	1.54	4.15	5.63	6.29
260	3.53	1.71	4.33	5.71	6.34
280	3.90	1.88	4.50	5.78	6.38
300	4.24	2.05	4.65	5.86	6.44

[a] See for example: *AIP Handbook* (McGraw-Hill, New York, 1963), p. 4–48; NBS Cryogenic Engr. Lab WADD Tech. Rep. 60-56, Part II (1960).

Table II. Molar specific heats at constant pressure, averaged over the temperature interval 77–295 K, for beryllium, graphite, silicon, and lead. The units are cal/mole K. Column 2 lists values measured in this experiment and column 3 lists values obtained from the data in Table I.

Sample	$\bar{C}_p$(exp)	$\bar{C}_p$(calc)
Be	2.32	2.10
C(graphite)	1.13	0.92
Si	3.31	3.29
Pb	6.00	6.18

Similar data were obtained for silicon and carbon. We used samples of 7.20 g of silicon and 9.66 g of carbon (graphite) and determined the specific heats to be 3.31 and 1.13 cal/mole K, respectively.

Dulong and Petit noted that the room temperature specific heats at constant pressure of most metals (except the lightest) were near $3R$ (24.94 J/mole K, or 5.96 cal/mole K). The value obtained for lead was near $3R$, but as expected the values obtained for the other elements were considerably smaller. It is instructive to give the students literature data for $C_p(T)$ and have them compute $\bar{C}_p$ by numerical integration over the appropriate temperature range. For convenience, sets of C_p and T data for the materials we have used (and also copper for reference) are given in Table I.

Table II shows the average specific heats that we measured compared to the values calculated from the data in Table I. The agreement is quite good for silicon, lead, and beryllium but not so good for carbon. It should be noted that the samples we used were all of very high purity. (We caution that some lead normally found around the laboratory may be of questionable purity and may give erroneous results.)

We also caution that students must be made aware of the hazards of improper handling of LN_2. In our laboratories, students are required to wear protective eye covers during the experiment. The laboratory instructor is responsible for the main LN_2 storage dewar and dispenses the nitrogen in small styrofoam cups to the students as required.

[1] Ohaus Dial-O-Gram model 310 balance. Ohaus Scale Corp., 29 Hanover Road, Florham Park, New Jersey 07932.

[2] Keithley Instruments, Inc., 28775 Aurora Road, Cleveland, Ohio 44139.

Remark on the isotropic form of the Schwarzschild interior metric

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In the course of determining the Schwarzschild interior solution in isotropic coordinates it turns out that to a given density and coordinate radius there in general correspond two possible spheres which have distinct masses and distinct physical radii. This situation is here described in detail.

I. INTRODUCTION

The well-known "Schwarzschild interior solution" is the metric (S_i) of a region of space-time occupied by a spherically symmetric static distribution of fluid the density $\rho(= T^4_{44}/8\pi)$ of which has a fixed positive value, that is, it is independent of position and time. No doubt it is considered so often in expositions of general relativity theory because of the ease with which its explicit form can be found. One should, however, bear in mind that it is formal in character in as far as the required constancy of ρ is unphysical: it implies that the speed of sound is infinite and causality is violated.[1] At any rate, an appropriate generic form of the metric is[2,3]

$$ds^2 = -e^\lambda \, d\bar{r}^2 - \bar{r}^2 \, e^\mu \, d\Omega^2 + e^\nu \, dt^2, \qquad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$ and λ, μ, ν are functions of $\bar{r}$ alone. One of these may be prescribed arbitrarily, subject to