Verifica della disuguaglianza di Cramer-Rao

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December 2024

1 Calcolo della matrice I

Considero un campione di valori (x_i, y_i) con i=1,..,n, suppongo di conoscere le x_i con un errore trascurabile, mentre le y_i suppongo siano indipendenti e seguano una distribuzione $N(\mu_i, \sigma_i^2)$, dove μ_i è dato da:

$$\mu_i = mx_i + q$$

La funzione di Likelihood è allora:

$$\mathscr{L}(y_1, y_2, ..., y_n; m, q) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_i} \exp\left\{-\sum_{i=1}^{n} \frac{(y_i - \mu_i)^2}{2\sigma_i^2}\right\}$$

Procedo calcolando le derivate rispetto ai paramteri e ottengo (per semplicità di notazione nelle sommatorie ometto il range del pedice i poichè è sempre lo stesso):

$$\frac{\partial \ln \mathscr{L}}{\partial m} = -\sum_i \frac{2(y_i - mx_i - q)}{2\sigma_i^2}(-x_i) = \sum_i \frac{x_i}{\sigma_i^2}(y_i - mx_i - q)$$

$$\frac{\partial^2 \ln \mathscr{L}}{\partial m^2} = -\sum_i \frac{x_i^2}{\sigma_i^2}$$

$$\frac{\partial \ln \mathscr{L}}{\partial q} = -\sum_i \frac{2(y_i - mx_i - q)}{2\sigma_i^2}(-1) = \sum_i \frac{y_i - mx_i - q}{\sigma_i^2}$$

$$\frac{\partial^2 \ln \mathscr{L}}{\partial m^2} = -\sum_i \frac{1}{\sigma_i^2}$$

$$\frac{\partial^2 \ln \mathscr{L}}{\partial m \partial q} = -\sum_i \frac{x_i}{\sigma_i^2}$$

Perciò la matrice I (definita come $I_{ij} = E[-\frac{\partial^2 \ln \mathscr{L}}{\partial \theta_i \partial \theta_j}]$) è:

$$\begin{pmatrix} \sum_i \frac{x_i^2}{\sigma_i^2} & \sum_i \frac{x_i}{\sigma_i^2} \\ \sum_i \frac{x_i}{\sigma_i^2} & \sum_i \frac{1}{\sigma_i^2} \end{pmatrix}$$

2 Stima dei paramteri

Per la stima dei parametri poniamo uguali a 0 le derivate prime del logaritmo della funzione di Likelihood:

$$\begin{cases} \sum_i \frac{x_i}{\sigma_i^2}(y_i - mx_i - q) = 0 \\ \sum_i \frac{y_i - mx_i - q}{\sigma_i^2} = 0 \end{cases}$$

Che diventa:

$$\begin{cases} \sum_i \frac{x_i}{\sigma_i^2} q = \sum_i \frac{x_i}{\sigma_i^2} y_i - \sum_i \frac{x_i^2}{\sigma_i^2} m \\ \sum_i \frac{1}{\sigma_i^2} q = \sum_i \frac{y_i}{\sigma_i^2} - \sum_i \frac{x_i}{\sigma_i^2} m \end{cases}$$

Isolo la q:

$$q = \frac{\sum_i \frac{y_i}{\sigma_i^2} - \sum_i \frac{x_i}{\sigma_i^2} m}{\sum_i \frac{1}{\sigma_i^2}}$$

Quindi

$$\sum_i \frac{x_i}{\sigma_i^2} \sum_i \frac{y_i}{\sigma_i^2} - \left(\sum_i \frac{x_i}{\sigma_i^2} \right)^2 = \sum_i \frac{x_i}{\sigma_i^2} y_i \sum_i \frac{1}{\sigma_i^2} - \sum_i \frac{x_i^2}{\sigma_i^2} \sum_i \frac{1}{\sigma_i^2}$$

E ottengo infine:

$$\tilde{m} = \frac{\sum_i \frac{x_i}{\sigma_i^2} y_i - \sum_i \frac{x_i}{\sigma_i^2} \sum_i \frac{y_i}{\sigma_i^2}}{\sum_i \frac{x_i^2}{\sigma_i^2} \sum_i \frac{1}{\sigma_i^2} - \left(\sum_i \frac{x_i}{\sigma_i^2} \right)^2}$$

Analogamente trovo:

$$\tilde{q} = \frac{\sum_i \frac{x_i^2}{\sigma_i^2} \sum_i \frac{y_i}{\sigma_i^2} - \sum_i \frac{x_i y_i}{\sigma_i^2} \sum_i \frac{x_i}{\sigma_i^2}}{\sum_i \frac{x_i^2}{\sigma_i^2} \sum_i \frac{1}{\sigma_i^2} - \left(\sum_i \frac{x_i}{\sigma_i^2} \right)^2}$$

Infine pongo:

$$D = \sum_i \frac{x_i^2}{\sigma_i^2} \sum_i \frac{1}{\sigma_i^2} - \left(\sum_i \frac{x_i}{\sigma_i^2} \right)^2$$

3 Calcolo della matrice delle covarianze

Per trovare le varianze e covarianze degli stimatori uso la propagazione delle varianze e covarianze:

$$\sigma_{\tilde{q}}^2 = \sum_i \left(\frac{\partial \tilde{q}}{\partial y_i} \right)^2 \sigma_i^2 = \sum_j \left[\frac{\partial}{\partial y_j} \left[\frac{\sum_i \frac{x_i}{\sigma_i^2} y_i - \sum_i \frac{x_i}{\sigma_i^2} \sum_i \frac{y_i}{\sigma_i^2}}{D} \right] \right]^2 \sigma_j^2 =$$

$$= \sum_j \left[\frac{\frac{1}{\sigma_j^2} \sum_i \frac{x_i^2}{\sigma_i^2} - \frac{x_j}{\sigma_j^2} \sum_i \frac{x_i}{\sigma_i^2}}{D} \right]^2 \sigma_j^2 = \sum_j \left[\frac{\sum_i \frac{x_i^2}{\sigma_i^2} - x_j \sum_i \frac{x_i}{\sigma_i^2}}{D} \right]^2 \frac{1}{\sigma_j^2} =$$

$$= \sum_j \frac{\left(\sum_i \frac{x_i^2}{\sigma_i^2} \right)^2 + x_j^2 \left(\sum_i \frac{x_i}{\sigma_i^2} \right)^2 - 2 x_j \sum_i \frac{x_i^2}{\sigma_i^2} \sum_i \frac{x_i}{\sigma_i^2}}{D^2} \frac{1}{\sigma_j^2} =$$

$$= \frac{\sum_i \frac{1}{\sigma_i^2} \left(\sum_i \frac{x_i^2}{\sigma_i^2} \right)^2 + \sum_i \frac{x_i^2}{\sigma_i^2} \left(\sum_i \frac{x_i}{\sigma_i^2} \right)^2 - 2 \sum_i \frac{x_i}{\sigma_i^2} \sum_i \frac{x_i^2}{\sigma_i^2} \sum_i \frac{x_i}{\sigma_i^2}}{D^2}$$

Nell'ultimo passaggio ho tenuto un unico indice per le due somme, proseguendo i calcoli si ottiene:

$$= \frac{\sum_i \frac{1}{\sigma_i^2} \left(\sum_i \frac{x_i^2}{\sigma_i^2} \right)^2 - \sum_i \frac{x_i^2}{\sigma_i^2} \left(\sum_i \frac{x_i}{\sigma_i^2} \right)}{D^2}$$

$$= \frac{\sum_i \frac{x_i^2}{\sigma_i^2}}{D}$$

Analogamente per $\tilde{m}$ si trova:

$$\sigma_{\tilde{m}}^2 = \frac{\sum_i \frac{1}{\sigma_i^2}}{D}$$

Per la covarianza invece:

$$cov(\tilde{m}; \tilde{q}) = \sum_j \frac{\partial \tilde{m}}{\partial y_j} \frac{\partial \tilde{q}}{\partial y_j} \sigma_j^2 = \sum_j \frac{\sigma_j^2}{D^2} \left[\left(\frac{x_j}{\sigma_j^2} \sum_i \frac{1}{\sigma_i^2} - \frac{1}{\sigma_j^2} \sum_i \frac{x_i}{\sigma_i^2} \right) \left(\frac{1}{\sigma_j^2} \sum_i \frac{x_i^2}{\sigma_i^2} - \frac{x_j}{\sigma_j^2} \sum_i \frac{x_i}{\sigma_i^2} \right) \right] =$$

$$\sum_j \frac{\sigma_j^2}{D^2} \left[\frac{x_j}{\sigma_j^4} \sum_i \frac{1}{\sigma_i^2} \sum_i \frac{x_i^2}{\sigma_i^2} - \frac{1}{\sigma_i^4} \sum_i \frac{x_i}{\sigma_i^2} \sum_i \frac{x_i^2}{\sigma_i^2} - \frac{x_j^2}{\sigma_i^4} \sum_i \frac{x_i}{\sigma_i^2} \sum_i \frac{1}{\sigma_i^2} + \frac{x_j}{\sigma_i^4} \sum_i \frac{x_i}{\sigma_i^2} \sum_i \frac{x_i}{\sigma_i^2} \right] =$$

$$= \frac{1}{D^2} \left[\sum_i \frac{x_i}{\sigma_i^2} \sum_i \frac{1}{\sigma_i^2} \sum_i \frac{x_i^2}{\sigma_i^2} - \sum_i \frac{1}{\sigma_i^2} \sum_i \frac{x_i}{\sigma_i^2} \sum_i \frac{x_i^2}{\sigma_i^2} - \sum_i \frac{x_i^2}{\sigma_i^2} \sum_i \frac{x_i}{\sigma_i^2} \sum_i \frac{1}{\sigma_i^2} + \sum_i \frac{x_i}{\sigma_i^2} \left(\sum_i \frac{x_i}{\sigma_i} \right)^2 \right] =$$

$$= -\frac{\sum_i \frac{x_i}{\sigma_i^2}}{D}$$

Perciò la matrice delle covarianze è:

$$\begin{pmatrix} \frac{\sum_i \frac{1}{\sigma_i^2}}{D} & \frac{\sum_i \frac{x_i}{\sigma_i^2}}{D} \\ \frac{\sum_i \frac{x_i}{\sigma_i^2}}{D} & \frac{\sum_i \frac{x_i^2}{\sigma_i^2}}{D} \end{pmatrix}$$

Mentre I era:

$$\begin{pmatrix} \sum_i \frac{x_i^2}{\sigma_i^2} & \sum_i \frac{x_i}{\sigma_i^2} \\ \sum_i \frac{x_i}{\sigma_i^2} & \sum_i \frac{1}{\sigma_i^2} \end{pmatrix}$$

4 Calcolo della matrice inversa e confronto

Per trovare l'inversa della matrice I uso il metodo di cramer:

$$detI = \sum_i \frac{x_i^2}{\sigma_i^2} \sum_i \frac{1}{\sigma_i^2} - \left(\sum_i \frac{x_i}{\sigma_i^2} \right)^2 = D$$

Perciò:

$$I^{-1} = \frac{1}{D} \begin{pmatrix} \frac{\sum_i \frac{1}{\sigma_i^2}}{D} & \frac{\sum_i \frac{x_i}{\sigma_i^2}}{D} \\ \frac{\sum_i \frac{x_i}{\sigma_i^2}}{D} & \frac{\sum_i \frac{x_i^2}{\sigma_i^2}}{D} \end{pmatrix} = V$$

Di conseguenza gli stimatori trovati per $\tilde{m}$ e $\tilde{q}$ sono degli stimatori efficienti a varianza minima, e quindi valgono le relazioni:

$$\begin{cases} \sigma_{\tilde{m}} = \frac{1}{D}\left(-\frac{\partial^2 \ln \mathscr{L}}{\partial q^2} \right) \\ \sigma_{\tilde{q}} = \frac{1}{D}\left(-\frac{\partial^2 \ln \mathscr{L}}{\partial m^2} \right) \\ cov(\tilde{m}, \tilde{q}) = \frac{1}{D}\left(\frac{\partial^2 \ln \mathscr{L}}{\partial m \partial q} \right) \end{cases}$$