Introduction to Bayesian Methods- 3

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Generalization to more than two discrete values:
the 1-of-K coding scheme and the Dirichlet pdf

Coin tosses and Bernoulli variates

Bernoulli variates:

$$x \sim \begin{cases} P(x=1) = \theta \\ P(x=0) = 1-\theta \end{cases} \quad \Rightarrow \quad \begin{aligned} \mathbb{E}(x) &= \theta \\ \mathrm{var}(x) &= \theta \end{aligned}$$

1-of-2 coding scheme for Bernoulli variates:

$$x \quad \Rightarrow \quad (1,0)^T \text{ OR } (0,1)^T$$

The Binomial variate as a sum of Bernoulli variates:

$$x_{\mathrm{Bin}} = \sum_{k=1}^{N} x_k \quad \Rightarrow \quad P(x_{\mathrm{Bin}}) = \frac{N!}{m!l!} x^m (1-x)^l$$

# of 1s

# of 0s

The case of multiple values

1-of-K coding scheme:

$$\mathbf{x} = (x_1, \ldots, x_k, \ldots, x_K)^T = (0, \ldots, 1, \ldots, 0)^T$$

Constraint:

$$\sum_{k=1}^{K} x_k = 1$$

Probability of each scalar component:

$$P(x_k = 1) = \theta_k; \quad \sum_{k=1}^{K} \theta_k = 1$$

Probability of a given vector:

$$P(\mathbf{x}|\boldsymbol{\theta}) = \prod_{k=1}^{K} \theta_k^{x_k}$$

The case of multiple values – 2

Dataset of N independent observations:

$$\mathbf{x}_n = (x_{n1}, \ldots, x_{nk}, \ldots, x_{nK})^T$$

Constraint:

$$\sum_{k=1}^{K} x_{nk} = 1$$

Probability of each scalar component:

$$P(x_{nk} = 1) = \theta_k; \quad \sum_{k=1}^{K} \theta_k = 1 \quad \Rightarrow \quad \mathbb{E}(\mathbf{x}|\boldsymbol{\theta}) = \boldsymbol{\theta}$$

Probability of a given vector:

$$P(\mathbf{x}_n|\boldsymbol{\theta}) = \prod_{k=1}^{K} \theta_k^{x_{nk}}$$

The case of multiple values – 3

Likelihood for a dataset of size N

$$p(\mathcal{D}|\boldsymbol{\theta}) = \prod_{n=1}^{N}\prod_{k=1}^{K}\theta_k^{x_{nk}} = \prod_{k=1}^{K}\theta_k^{\sum_n x_{nk}} = \prod_{k=1}^{K}\theta_k^{m_k}$$

where

$$m_k = \sum_n x_{nk}$$

which is the number of observations of the k-th element.

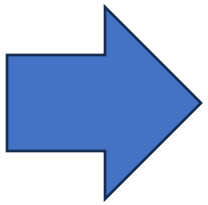
To maximize the log-likelihood we must use a Lagrange multiplier

$$\frac{\partial}{\partial\theta_j}\left[\sum_{k=1}^{K} m_k \ln\theta_k - \lambda\left(\sum_k \theta_k - 1\right)\right] = \frac{m_j}{\theta_j} - \lambda = 0 \quad \Rightarrow \quad \theta_j = \frac{m_j}{\lambda}$$

The case of multiple values – 4

From the normalization condition

$$1 = \sum_{j=1}^{K} \theta_j = \sum_{j=1}^{K} \frac{m_j}{\lambda} = \frac{N}{\lambda} \quad \Rightarrow \quad \lambda = N$$



$$\hat{\theta}_j = \frac{m_j}{N}$$

We do not need ALL the data to determine the parameters, we only need the m_j's, which are thus an example of **sufficient statistic**

The case of multiple values – 5

Except for the normalization, the likelihood

$$p(\mathcal{D}|\boldsymbol{\theta}) = \prod_{k=1}^{K} \theta_k^{m_k}$$

is proportional to the multinomial distribution of the quantities m_j's

$$\text{Mult}(m_1, \ldots, m_K|\boldsymbol{\theta}) = \frac{N!}{\prod_{k=1}^{K} m_k!} \prod_{k=1}^{K} \theta_k^{m_k}$$

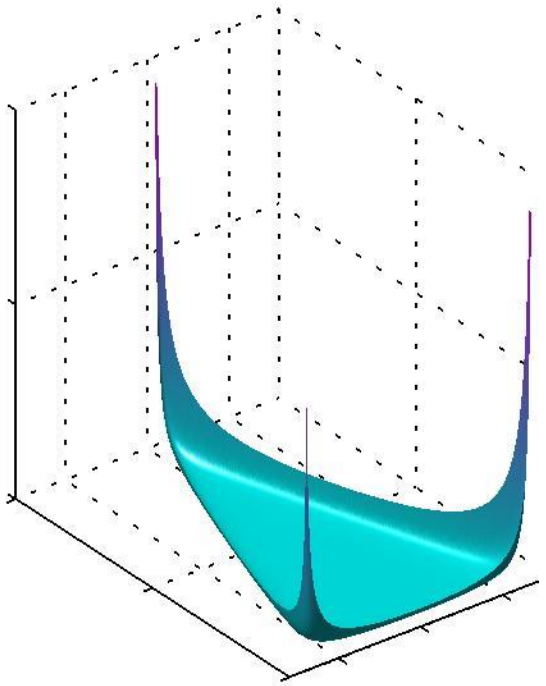
and the corresponding conjugate distribution is the Dirichlet distribution (a generalization of the Beta distribution)

$$\text{Dir}(\boldsymbol{\theta}|\mathbf{m}) = \frac{\Gamma[m_0]}{\prod_{k=1}^{K} \Gamma[m_k]} \prod_{k=1}^{K} \theta_k^{m_k} \quad \text{with } m_0 = \sum_k m_k$$

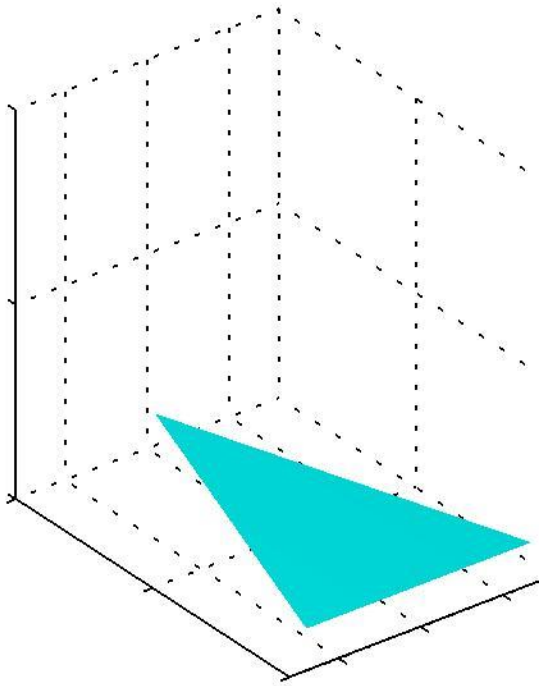
Thus, the Dirichlet distribution is the conjugate to the posterior and we can use it as a prior just as the Beta distribution of the previous example.

$$\mathrm{Dir}(\boldsymbol{\theta}|\boldsymbol{\alpha}) = \frac{\Gamma[\alpha_0]}{\prod_{k=1}^{K}\Gamma[\alpha_k]}\prod_{k=1}^{K}\theta_k^{\alpha_k} \quad \text{with } \alpha_0 = \sum_k \alpha_k$$

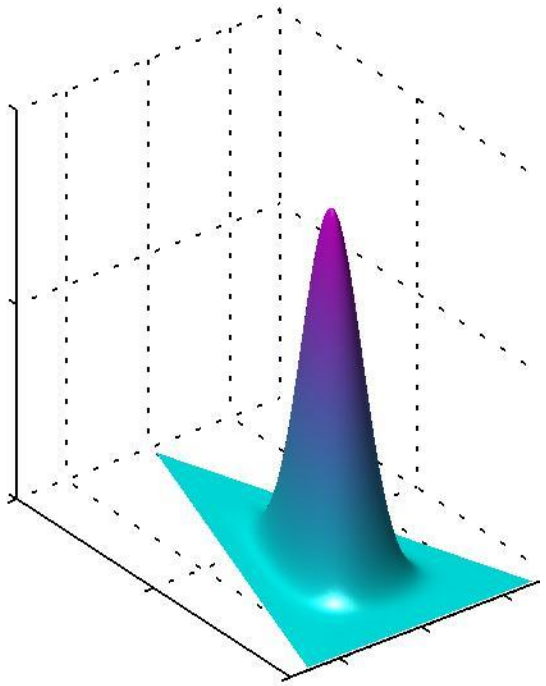
Dirichlet prior



$\alpha_k = 0.1$



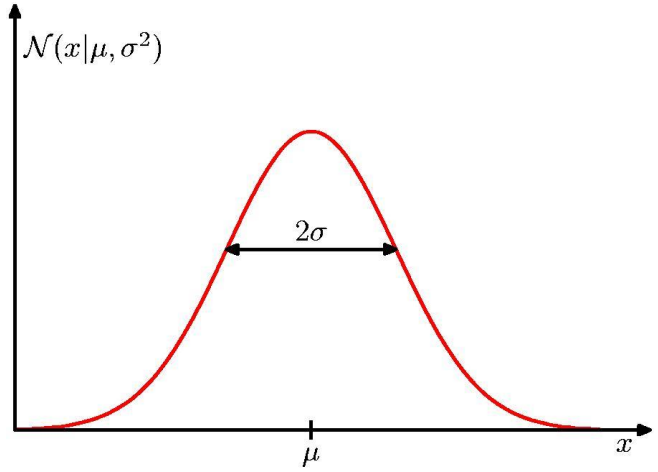
$\alpha_k = 1$



$\alpha_k = 10$

The (multivariate) Gaussian distribution:

1-dimensional Gaussian distribution



Mean

Variance

$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$$

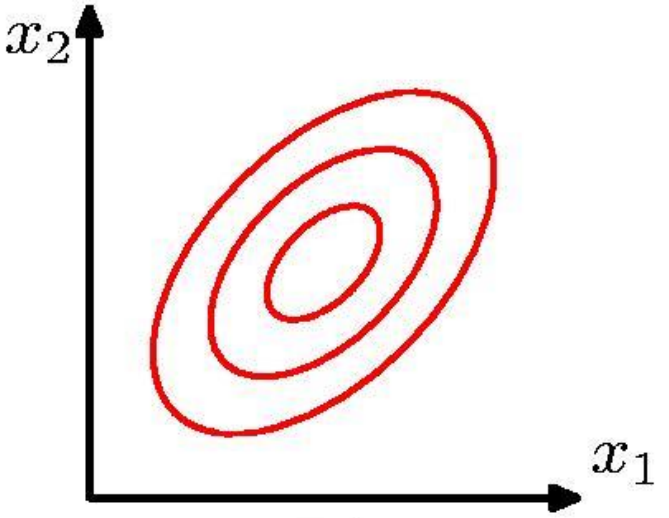
D-dimensional Gaussian distribution

Vector of means

Covariance matrix

$$\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

Square root of the determinant of the covariance matrix



$$\mathcal{N}(\mathbf{x}|\boldsymbol{\mu},\boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}}\frac{1}{|\boldsymbol{\Sigma}|^{1/2}}\exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

$$\Delta^2 = (\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$$

Mahalanobis distance

$$\boldsymbol{\Sigma}\mathbf{u}_i = \lambda_i\mathbf{u}_i$$

Eingenvalue equation for the **covariance matrix**

$$\boldsymbol{\Sigma}^{-1} = \sum_{i=1}^{D}\frac{1}{\lambda_i}\mathbf{u}_i\mathbf{u}_i^{\mathrm{T}}$$

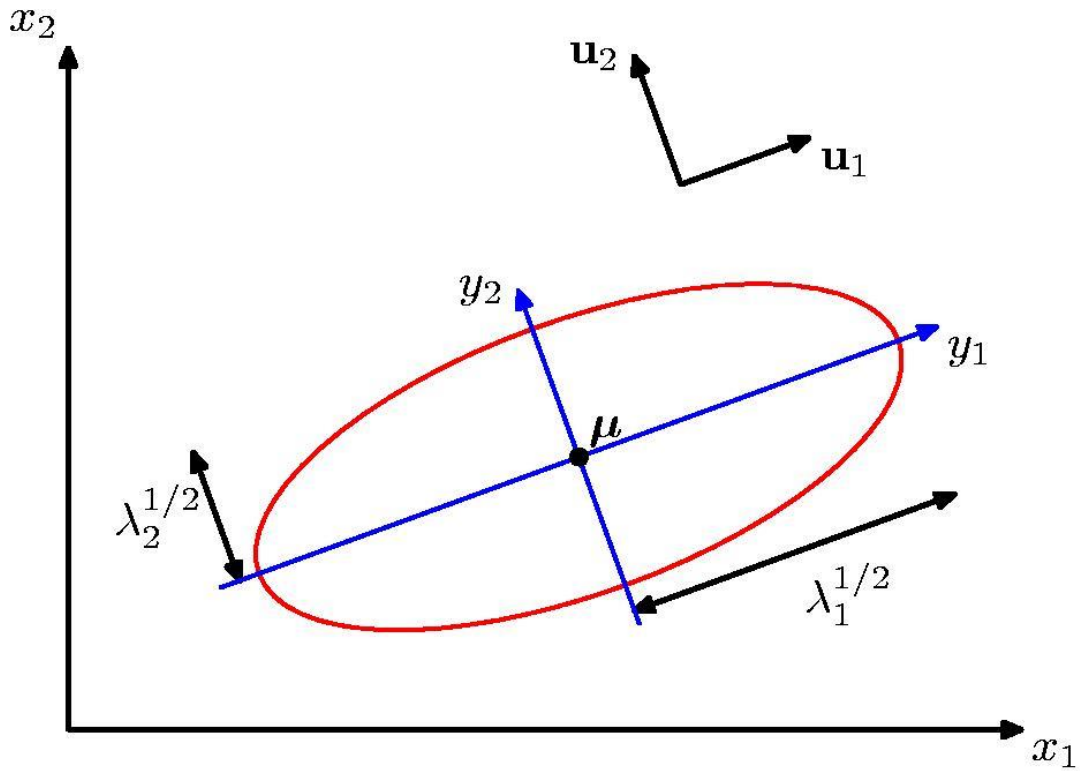
Diagonal form of the **precision matrix**

$$\Delta^2 = \sum_{i=1}^{D}\frac{y_i^2}{\lambda_i}$$

Mahalanobis distance in the rotated/translated reference system

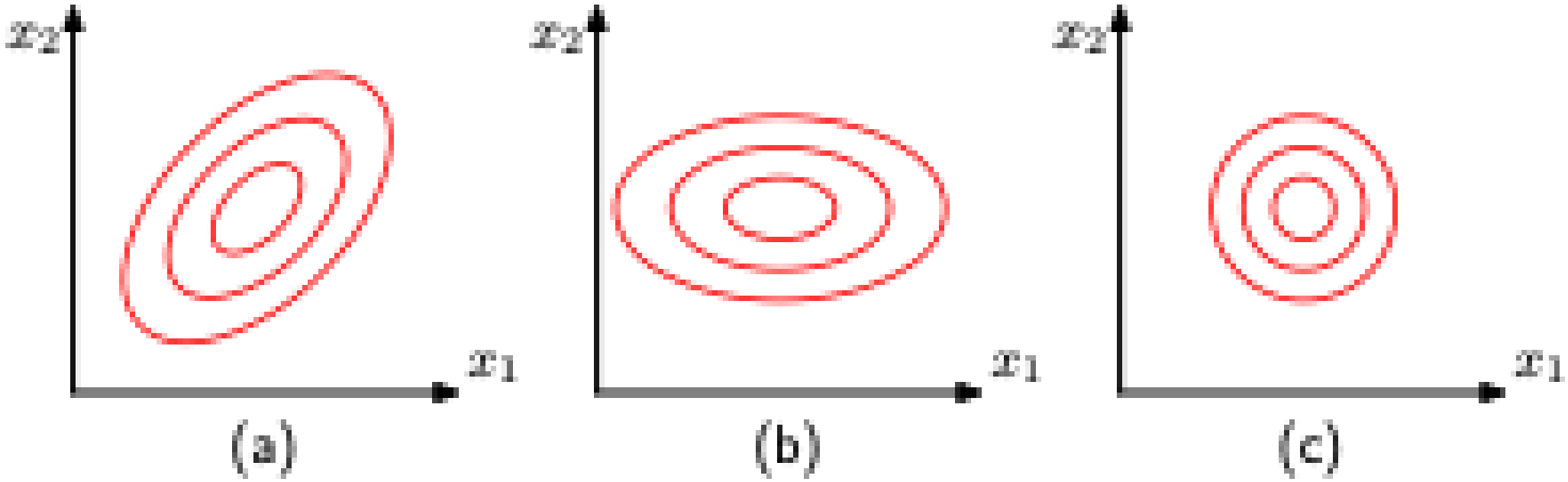
$$y_i = \mathbf{u}_i^{\mathrm{T}}(\mathbf{x}-\boldsymbol{\mu})$$

Rotated/translated coordinates



$$p(\mathbf{y}) = \prod_{j=1}^{D}\frac{1}{(2\pi\lambda)^{1/2}}\exp\left(-\frac{y_j^2}{2\lambda_j}\right)$$

pdf in the rotated/translated system



Contours of constant probability density for a Gaussian distribution in two dimensions in which the covariance matrix is (a) of general form, (b) diagonal, in which the elliptical contours are aligned with the coordinate axes, and (c) proportional to the identity matrix, in which the contours are concentric circles.

Conditional Gaussian distribution/Marginalization

An important property of the multivariate Gaussian distribution is that if two sets of variables are jointly Gaussian, then the conditional distribution of one set conditioned on the other is again Gaussian. Similarly, the marginal distribution of either set is also Gaussian.

Here, we assume that the a set contains p variables and the b set contains q variables, so that the partitioned matrix contains one p x p matrix and one p x q matrix in the first row, and one q x p matrix and a q x q matrix in the second row.

$$p(\mathbf{x}) = \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_a \\ \mathbf{x}_b \end{pmatrix} \qquad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_a \\ \boldsymbol{\mu}_b \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{aa} & \boldsymbol{\Sigma}_{ab} \\ \boldsymbol{\Sigma}_{ba} & \boldsymbol{\Sigma}_{bb} \end{pmatrix}$$

$$\boldsymbol{\Lambda} \equiv \boldsymbol{\Sigma}^{-1} \qquad \boldsymbol{\Lambda} = \begin{pmatrix} \boldsymbol{\Lambda}_{aa} & \boldsymbol{\Lambda}_{ab} \\ \boldsymbol{\Lambda}_{ba} & \boldsymbol{\Lambda}_{bb} \end{pmatrix}$$

Sometimes it is more convenient to work with the precision matrix. This is the partitioned form of the precision matrix.
Note that in general the off-diagonal terms are not square matrices.

Conditional Gaussian distribution/Marginalization – "Completing the square"

Now consider the following

$$p(\mathbf{x}_a|\mathbf{x}_b) \longrightarrow -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})$$
$$= -\frac{1}{2}(\mathbf{x}_a-\boldsymbol{\mu}_a)^T\Lambda_{aa}(\mathbf{x}_a-\boldsymbol{\mu}_a) - \frac{1}{2}(\mathbf{x}_a-\boldsymbol{\mu}_a)^T\Lambda_{ab}(\mathbf{x}_b-\boldsymbol{\mu}_b)$$
$$-\frac{1}{2}(\mathbf{x}_b-\boldsymbol{\mu}_b)^T\Lambda_{ba}(\mathbf{x}_a-\boldsymbol{\mu}_a) - \frac{1}{2}(\mathbf{x}_b-\boldsymbol{\mu}_b)^T\Lambda_{bb}(\mathbf{x}_b-\boldsymbol{\mu}_b)$$

Quadratic term in a after fixing the b part

Linear terms in a after fixing the b part

Constant term in a after fixing the b part

and

$$p(\mathbf{x}) \longrightarrow -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}) = \underbrace{-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x}}_{\text{Quadratic term}} + \underbrace{\mathbf{x}^T\Sigma^{-1}\boldsymbol{\mu}}_{\text{Linear term}} + \text{ const}$$

Conditional Gaussian distribution/Marginalization – "Completing the square" – 2

$$
\begin{aligned}
p(\mathbf{x}_a|\mathbf{x}_b) \longrightarrow & -\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu}) \\
&= -\frac{1}{2}(\mathbf{x}_a-\boldsymbol{\mu}_a)^T\Lambda_{aa}(\mathbf{x}_a-\boldsymbol{\mu}_a) - \frac{1}{2}(\mathbf{x}_a-\boldsymbol{\mu}_a)^T\Lambda_{ab}(\mathbf{x}_b-\boldsymbol{\mu}_b) \\
&\quad -\frac{1}{2}(\mathbf{x}_b-\boldsymbol{\mu}_b)^T\Lambda_{ba}(\mathbf{x}_a-\boldsymbol{\mu}_a) - \frac{1}{2}(\mathbf{x}_b-\boldsymbol{\mu}_b)^T\Lambda_{bb}(\mathbf{x}_b-\boldsymbol{\mu}_b) \\
&= -\frac{1}{2}\mathbf{x}_a^T\Lambda_{aa}\mathbf{x}_a + \mathbf{x}_a^T\left[\Lambda_{aa}\boldsymbol{\mu}_a - \Lambda_{ab}(\mathbf{x}_b-\boldsymbol{\mu}_b)\right] + \text{ const} \qquad (*)
\end{aligned}
$$

$$
p(\mathbf{x}_a) \longrightarrow -\frac{1}{2}(\mathbf{x}_a-\boldsymbol{\mu}_{a|b})^T\Sigma_{a|b}^{-1}(\mathbf{x}_a-\boldsymbol{\mu}_{a|b}) = -\frac{1}{2}\mathbf{x}^T\Sigma_{a|b}^{-1}\mathbf{x} + \mathbf{x}_a^T\Sigma_{a|b}^{-1}\boldsymbol{\mu}_{a|b} + \text{const}
$$

Then, by comparing the expressions we find

$$
\begin{aligned}
\Sigma_{a|b}^{-1} &= \Lambda_{aa} \\
\Sigma_{a|b}^{-1}\boldsymbol{\mu}_{a|b} &= \Lambda_{aa}\boldsymbol{\mu}_a - \Lambda_{ab}(\mathbf{x}_b-\boldsymbol{\mu}_b)
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_{a|b}^{-1} &= \Lambda_{aa} \\
\boldsymbol{\mu}_{a|b} &= \boldsymbol{\mu}_a - \Lambda_{aa}^{-1}\Lambda_{ab}(\mathbf{x}_b-\boldsymbol{\mu}_b)
\end{aligned}
$$

Conditional Gaussian distribution/Marginalization – "Completing the square" – 3

General result on partitioned matrices (see also https://en.wikipedia.org/wiki/Schur_complement)

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}$$

$$\begin{pmatrix} \Sigma_{aa} & \Sigma_{ab} \\ \Sigma_{ba} & \Sigma_{bb} \end{pmatrix}^{-1} = \begin{pmatrix} \Lambda_{aa} & \Lambda_{ab} \\ \Lambda_{ba} & \Lambda_{bb} \end{pmatrix}$$

$$= \begin{pmatrix} (\Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba})^{-1} & -(\Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba})^{-1}\Sigma_{ab}\Sigma_{bb}^{-1} \\ -(\Sigma_{bb} - \Sigma_{ba}\Sigma_{aa}^{-1}\Sigma_{ab})^{-1}\Sigma_{ba}\Sigma_{aa}^{-1} & (\Sigma_{bb} - \Sigma_{ba}\Sigma_{aa}^{-1}\Sigma_{ab})^{-1} \end{pmatrix}$$

$$\Sigma_{a|b} = \Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba}$$

$$\boldsymbol{\mu}_{a|b} = \boldsymbol{\mu}_a + \Sigma_{ab}\Sigma_{bb}^{-1}(\mathbf{x}_b - \boldsymbol{\mu}_b)$$

Conditional Gaussian distribution/Marginalization – Marginalization with respect to x_b

$$p(\mathbf{x}_a) = \int p(\mathbf{x}_a, \mathbf{x}_b)\, d\mathbf{x}_b$$ Marginalized distribution

We use the previous results and pick terms quadratic and linear in $\mathbf{x}_b$ (see *)

$$-\frac{1}{2}\mathbf{x}_b^T \mathbf{\Lambda}_{bb}\mathbf{x}_b + \mathbf{x}_b^T \mathbf{m} = -\frac{1}{2}(\mathbf{x}_b - \mathbf{\Lambda}_{bb}^{-1}\mathbf{m})^T \mathbf{\Lambda}_{bb}(\mathbf{x}_b - \mathbf{\Lambda}_{bb}^{-1}\mathbf{m}) + \frac{1}{2}\mathbf{m}^T \mathbf{\Lambda}_{bb}^{-1}\mathbf{m}$$

where

$$\mathbf{m} = \mathbf{\Lambda}_{bb}\boldsymbol{\mu}_b - \mathbf{\Lambda}_{ba}(\mathbf{x}_a - \boldsymbol{\mu}_a)$$

Then, integrating the first term we obtain a standard normalization factor (which does not depend on $\mathbf{x}_a$) times a normal distribution with exponent

$$\frac{1}{2}\left[\mathbf{\Lambda}_{bb}\boldsymbol{\mu}_b - \mathbf{\Lambda}_{ba}(\mathbf{x}_a - \boldsymbol{\mu}_a)\right]^T \mathbf{\Lambda}_{bb}^{-1}\left[\mathbf{\Lambda}_{bb}\boldsymbol{\mu}_b - \mathbf{\Lambda}_{ba}(\mathbf{x}_a - \boldsymbol{\mu}_a)\right]$$

a-dependent exponent after integrating over the b part (the constant in the b part)

$$-\frac{1}{2}\mathbf{x}_a^T \mathbf{\Lambda}_{aa}\mathbf{x}_a + \mathbf{x}_a^T(\mathbf{\Lambda}_{aa}\boldsymbol{\mu}_a + \mathbf{\Lambda}_{ab}\boldsymbol{\mu}_b) + \text{const}$$

previous quadratic and linear terms

Conditional Gaussian distribution/Marginalization – Marginalization with respect to x_b – 2

Expanding and simplifying we find

$$\frac{1}{2}[\Lambda_{bb}\boldsymbol{\mu}_b - \Lambda_{ba}(\mathbf{x}_a - \boldsymbol{\mu}_a)]^T\Lambda_{bb}^{-1}[\Lambda_{bb}\boldsymbol{\mu}_b - \Lambda_{ba}(\mathbf{x}_a - \boldsymbol{\mu}_a)] - \frac{1}{2}\mathbf{x}_a^T(\Lambda_{aa} - \Lambda_{ab}\Lambda_{bb}^{-1}\Lambda_{ba})\mathbf{x}_a + \text{const}$$

$$= -\frac{1}{2}\mathbf{x}_a^T(\Lambda_{aa} - \Lambda_{ab}\Lambda_{bb}^{-1}\Lambda_{ba})\mathbf{x}_a + \mathbf{x}_a^T(\Lambda_{aa} - \Lambda_{ab}\Lambda_{bb}^{-1}\Lambda_{ba})\boldsymbol{\mu}_a + \text{const}$$

and by comparison with

$$p(\mathbf{x}) \longrightarrow -\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu}) = -\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x} + \mathbf{x}^T\Sigma^{-1}\boldsymbol{\mu} + \text{ const}$$

we find (the mean is unchanged by the marginalization integral)

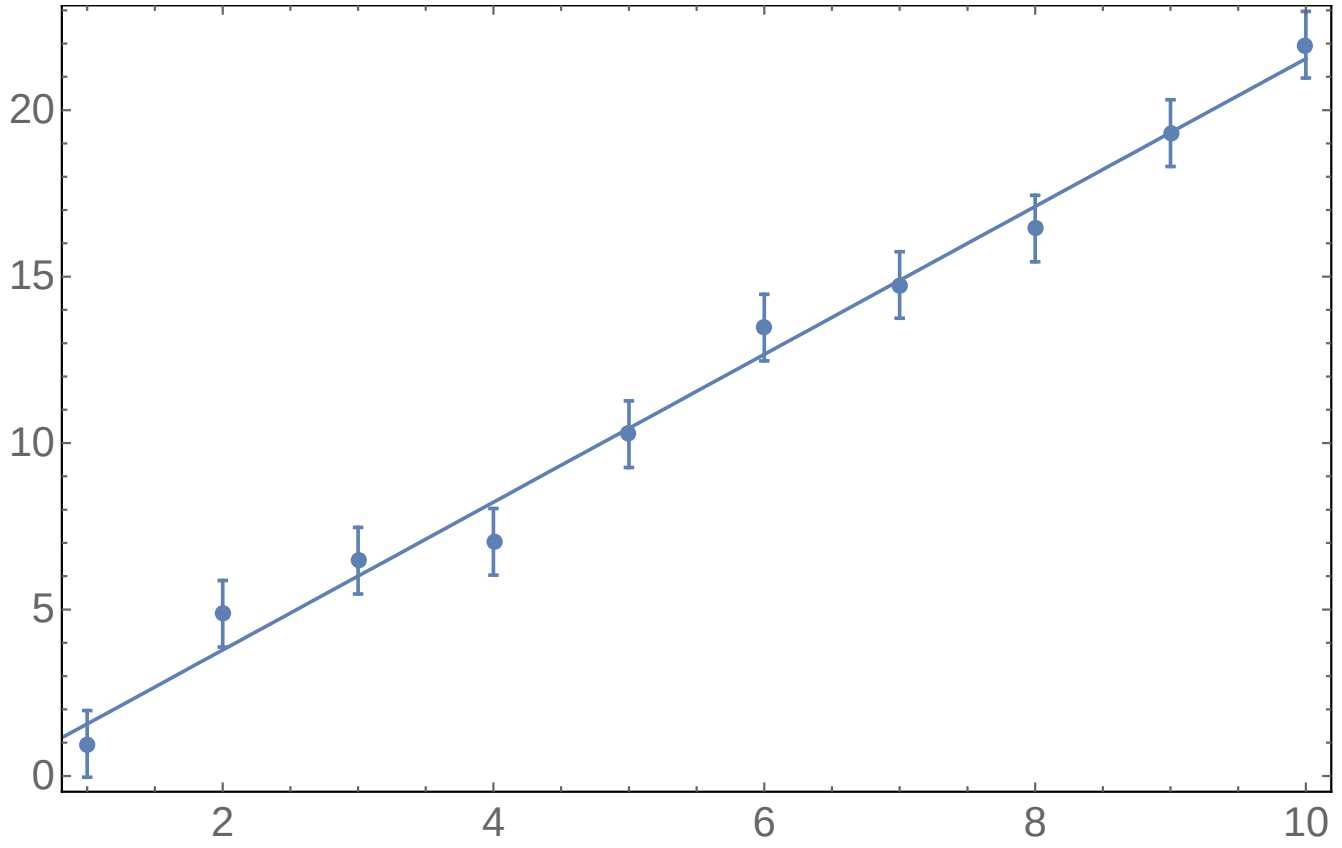
$$\Sigma_a = \Lambda_{aa} - \Lambda_{ab}\Lambda_{bb}^{-1}\Lambda_{ba}$$

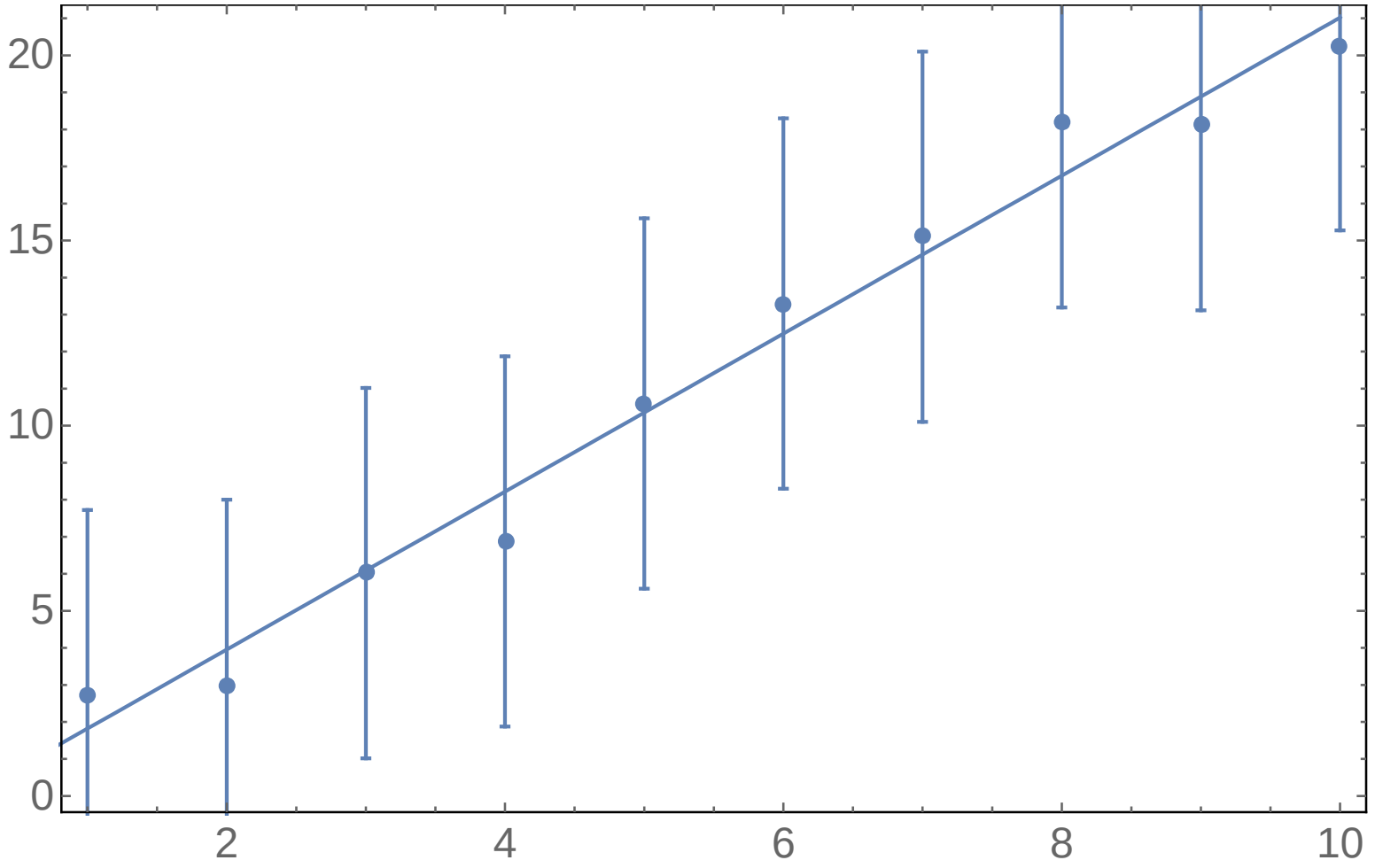
$$\boldsymbol{\mu}_a = \boldsymbol{\mu}_a$$

Example of Bayesian estimate using objective priors: uncalibrated Gaussian measurement uncertainties

Here, we consider the case where we must find the mean value with given measurement uncertainties that are systematically multiplied by an unknown scale factor, under the assumption of Gaussianity.

In this example we "complete the square" with "old style" methods, to provide a comparison with the methods developed earlier.




20
15
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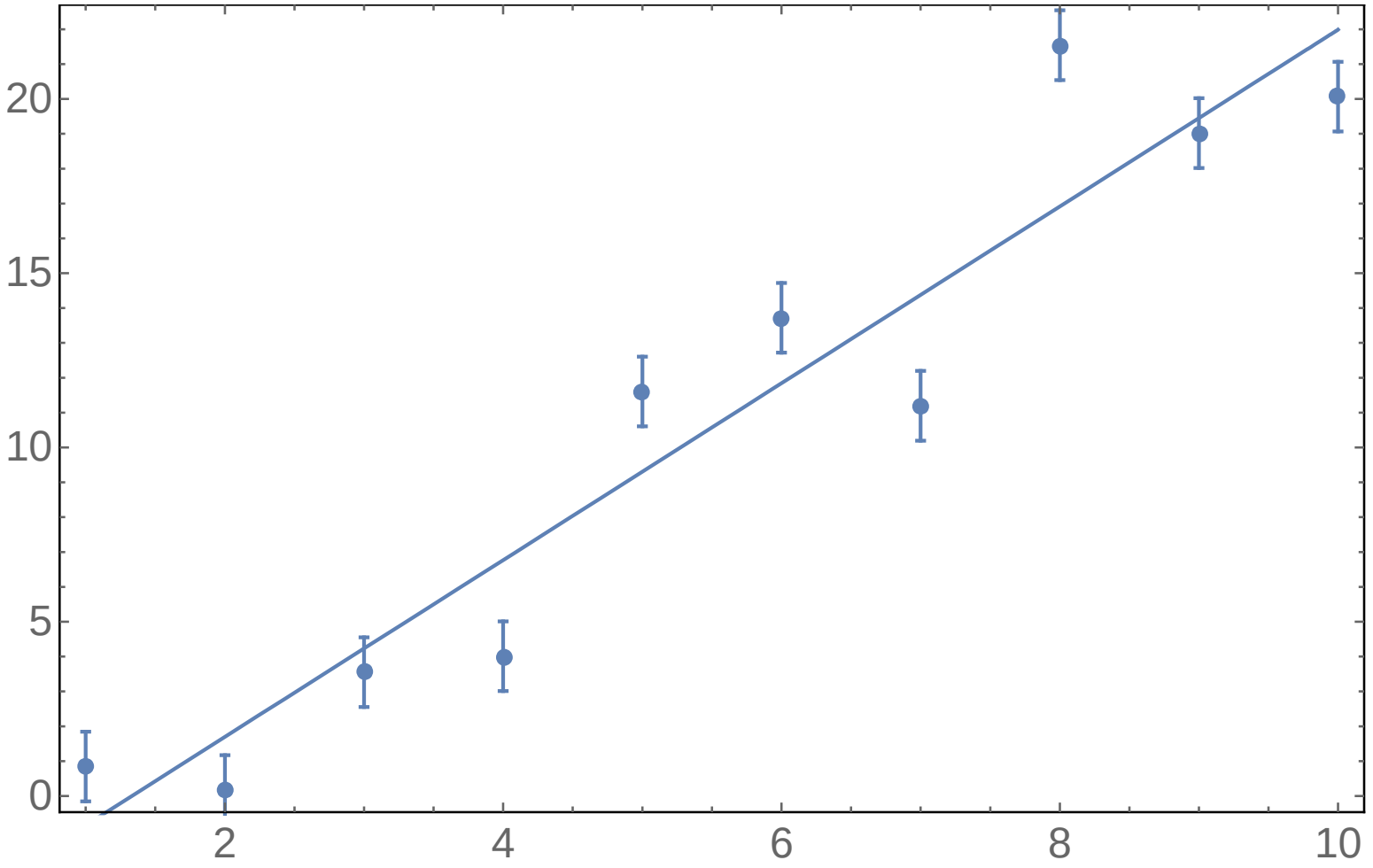
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The likelihood has a Gaussian structure

$$P(\mathbf{d} \mid \mu, \boldsymbol{\sigma}, \alpha) = \prod_{k=1}^{N} \frac{1}{\sqrt{2\pi\alpha^2\sigma_k^2}} \exp\left[-\frac{(d_k - \mu)^2}{2\alpha^2\sigma_k^2} \right]$$

$$= \frac{1}{(2\pi)^{N/2}\,\alpha^N} \left(\prod_{k=1}^{N} \frac{1}{\sigma_k} \right) \exp\left[-\frac{1}{2\alpha^2} \sum_{k=1}^{N} \frac{(d_k - \mu)^2}{\sigma_k^2} \right]$$

we must rearrange the exponent as usual ...

$$\sum_{k=1}^{N} \frac{(d_k - \mu)^2)}{\sigma_k^2} = \sum_{k=1}^{N} \frac{d_k^2}{\sigma_k^2} - 2\mu \sum_{k=1}^{N} \frac{d_k}{\sigma_k^2} + \mu^2 \sum_{k=1}^{N} \frac{1}{\sigma_k^2} = \frac{ND}{\sigma_M^2} - 2\mu \frac{NM}{\sigma_M^2} + \mu^2 \frac{1}{\sigma_M^2}$$

$$= \frac{N}{\sigma_M^2} \left(D - 2\mu M + \mu^2 \right)$$

$$\text{where } \frac{1}{\sigma_M^2} = \frac{1}{N} \sum_{k=1}^{N} \frac{1}{\sigma_k^2}; \quad M = \sum_{k=1}^{N} \frac{d_k}{\sigma_k^2} \Big/ \sum_{k=1}^{N} \frac{1}{\sigma_k^2}; \quad D = \sum_{k=1}^{N} \frac{d_k^2}{\sigma_k^2} \Big/ \sum_{k=1}^{N} \frac{1}{\sigma_k^2}$$

therefore, the likelihood is

$$P(\mathbf{d}|\mu, \sigma, \alpha) = \frac{1}{(2\pi)^{N/2} \alpha^N} \left(\prod_{k=1}^{N} \frac{1}{\sigma_k} \right) \exp \left[-\frac{N}{2\alpha^2 \sigma_M^2} (D - 2\mu M + \mu^2) \right]$$

Now we estimate the scale factor from Bayes' theorem

$$p(\alpha|\mathbf{d},\boldsymbol{\sigma}) = \frac{p(\mathbf{d}|\alpha,\boldsymbol{\sigma})}{\int_\alpha p(\mathbf{d}|\alpha',\boldsymbol{\sigma})p(\alpha')d\alpha'}p(\alpha)$$

however, we need first to marginalize the likelihood with respect to the mean, which in this case is a ***nuisance parameter***

we take a uniform prior for the mean (a Jeffrey's prior, see later)

$$P(\mathbf{d}\,|\,\boldsymbol{\sigma},\alpha) = \int_\mu P(\mathbf{d}\,|\,\mu,\boldsymbol{\sigma},\alpha)P(\mu|\boldsymbol{\sigma},\alpha)d\mu$$

$$= \frac{1}{W}\int_{\mu_{\min}}^{\mu_{\max}} P(\mathbf{d}\,|\,\mu,\boldsymbol{\sigma},\alpha)d\mu$$

$$\approx \frac{1}{W}\frac{1}{(2\pi)^{N/2}\alpha^N}\left(\prod_{k=1}^{N}\frac{1}{\sigma_k}\right)\int_{-\infty}^{+\infty}\exp\left[-\frac{N}{2\alpha^2\sigma_M^2}\left(D-2\mu M+\mu^2\right)\right]d\mu$$

$$W = \mu_{\max} - \mu_{\min}$$

as usual ...

$$D - 2\mu M + \mu^2 = \mu^2 - 2\mu M + M^2 + D - M^2 = (\mu - M)^2 + D - M^2$$

... therefore, the marginalized likelihood is:

$$P(\mathbf{d} \mid \boldsymbol{\sigma}, \alpha) \approx \frac{1}{W} \frac{1}{(2\pi)^{N/2} \alpha^N} \left(\prod_{k=1}^{N} \frac{1}{\sigma_k} \right) \int_{-\infty}^{+\infty} \exp\left\{ -\frac{N}{2\alpha^2 \sigma_M^2} \left[(\mu - M)^2 + D - M^2 \right] \right\} d\mu$$

$$= \frac{1}{W} \frac{1}{(2\pi)^{N/2} \alpha^N} \left(\prod_{k=1}^{N} \frac{1}{\sigma_k} \right) \exp\left(-\frac{N\left(D - M^2\right)}{2\alpha^2 \sigma_M^2} \right) \sqrt{\frac{2\pi\alpha^2 \sigma_M^2}{N}}$$

$$p(\alpha|\mathbf{d},\boldsymbol{\sigma}) = \frac{p(\mathbf{d}|\alpha,\boldsymbol{\sigma})}{\int_\alpha p(\mathbf{d}|\alpha',\boldsymbol{\sigma})p(\alpha')d\alpha'}p(\alpha)$$

$$= \frac{\frac{1}{\alpha^{N-1}}\exp\left(-\frac{N(D-M^2)}{2\alpha^2\sigma_M^2}\right)}{\int_\alpha \frac{1}{\alpha'^{N-1}}\exp\left(-\frac{N(D-M^2)}{2\alpha'^2\sigma_M^2}\right)p(\alpha')d\alpha'}p(\alpha)$$

$$P(\alpha) \propto \frac{1}{\alpha}$$

for the standard deviation we take again a Jeffreys' prior

$$p(\alpha|\mathbf{d},\boldsymbol{\sigma}) = \frac{\frac{1}{\alpha^{N-1}}\exp\left(-\frac{N(D-M^2)}{2\alpha^2\sigma_M^2}\right)\frac{1}{\alpha}}{\int_\alpha \frac{1}{\alpha'^{N-1}}\exp\left(-\frac{N(D-M^2)}{2\alpha'^2\sigma_M^2}\right)\frac{1}{\alpha'}d\alpha'}; \quad A^2 = \frac{N(D-M^2)}{2\sigma_M^2}$$

$$\Rightarrow \quad p(\alpha|\mathbf{d},\boldsymbol{\sigma}) \rightarrow \frac{\frac{1}{\alpha^N}\exp\left(-\frac{A^2}{\alpha^2}\right)}{\int_0^\infty \frac{1}{\alpha'^N}\exp\left(-\frac{A^2}{\alpha'^2}\right)d\alpha'}$$

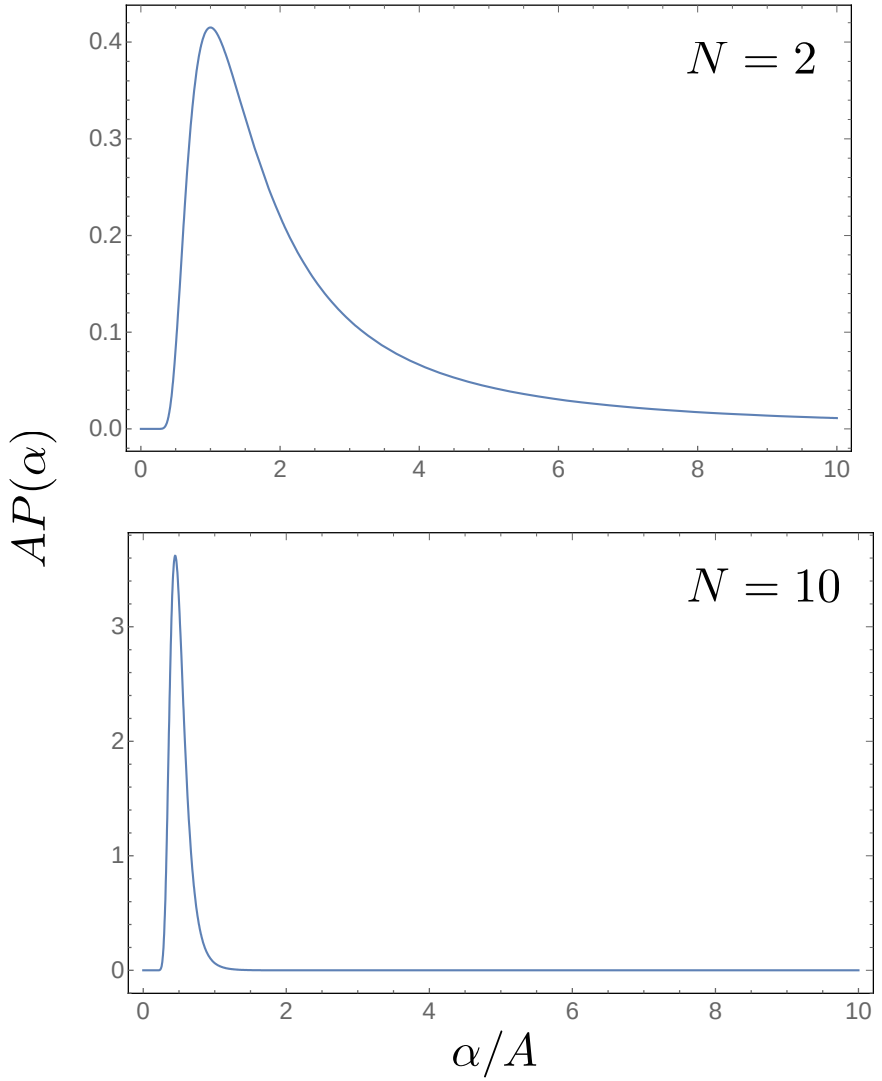
evaluation of $\int_0^\infty \frac{1}{\alpha'^N} \exp\left(-\frac{A^2}{\alpha'^2}\right) d\alpha'$

$$\frac{A^2}{\alpha^2} = x; \quad \alpha = \frac{A}{\sqrt{x}}; \quad d\alpha = -\frac{A}{2x^{3/2}} dx$$

$$\int_0^\infty \frac{x^{N/2}}{A^N} \exp(-x) \frac{A}{2x^{3/2}} dx = \frac{1}{2A^{N-1}} \int_0^\infty x^{\frac{N-1}{2}-1} \exp(-x) dx = \frac{1}{2A^{N-1}} \Gamma\left(\frac{N-1}{2}\right)$$

$$p(\alpha|\mathbf{d}, \boldsymbol{\sigma}) \to \frac{\frac{2A^{N-1}}{\alpha^N} \exp\left(-\frac{A^2}{\alpha^2}\right)}{\Gamma\left(\frac{N-1}{2}\right)}$$

$$P(\alpha|\mathbf{d},\boldsymbol{\sigma}) = \frac{(2A^{N-1}/\alpha^N)\exp(-A^2/\alpha^2)}{\Gamma\left[(N-1)/2\right]}$$

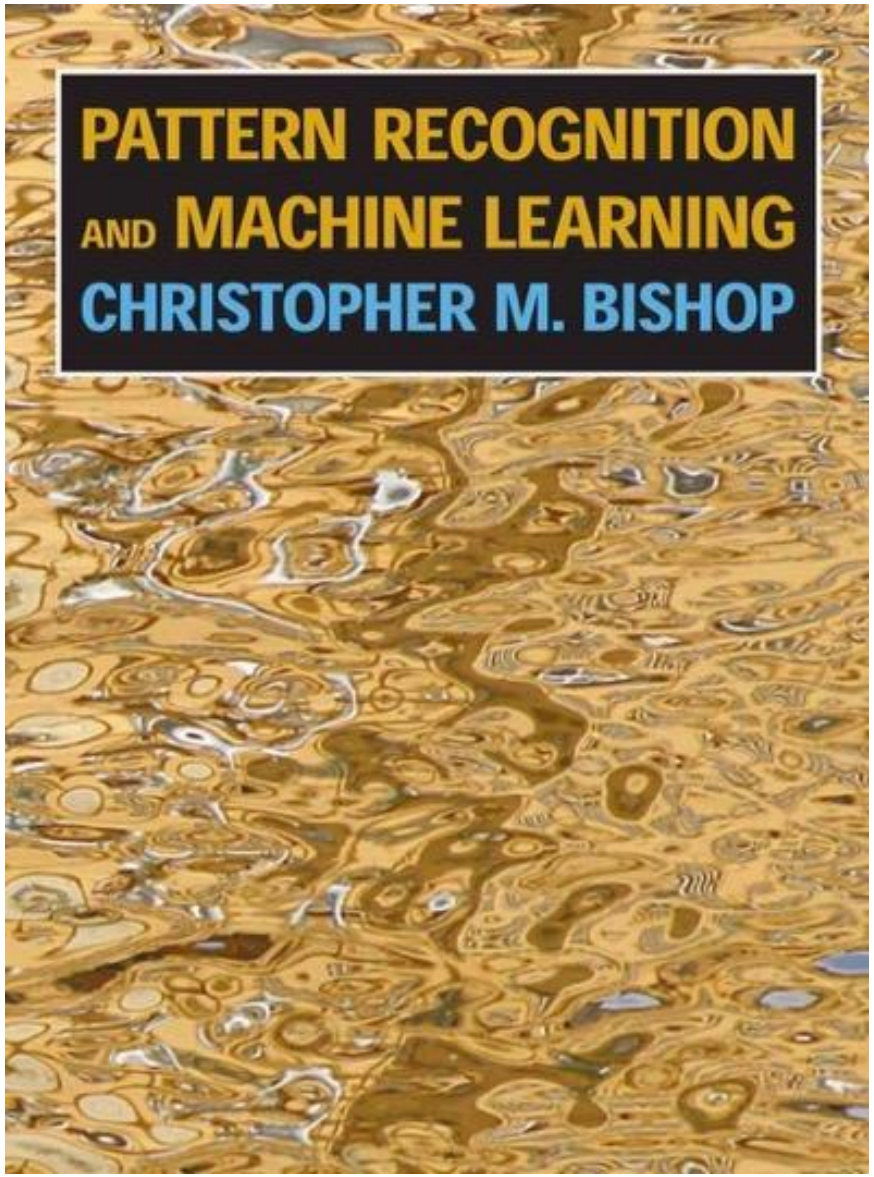


we take the MAP estimate of the scale parameter from the pdf

$$p(\alpha|\mathbf{d},\boldsymbol{\sigma}) = \frac{\dfrac{2A^{N-1}}{\alpha^N}\exp\left(-\dfrac{A^2}{\alpha^2}\right)}{\Gamma\left(\dfrac{N-1}{2}\right)}$$

$$\frac{d}{d\alpha}P(\alpha\,|\,\mathbf{d},\boldsymbol{\sigma}) \propto -\frac{N}{\alpha^{N+1}}\exp\left(-\frac{A^2}{\alpha^2}\right)+\frac{2A^2}{\alpha^{N+3}}\exp\left(-\frac{A^2}{\alpha^2}\right)=0$$

$$\Rightarrow \quad N\alpha^2 = 2A^2 \quad \Rightarrow \quad \alpha_{\mathrm{MAP}} = \sqrt{\frac{2}{N}}A = \sqrt{\frac{D-M^2}{\sigma_M^2}}$$



Part of the material in these slides is taken from this book.

The book is freely downloadable, see the website

https://www.microsoft.com/en-us/research/people/cmbishop/prml-book/