Special Relativity and notation

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1 Basics

Special Relativity (SR) is a necessary prerequisite of this course and is taken for granted. In this initial chapter I set out the conventions that are followed in the lecture notes.

1. For clarity, I always include all the physical constant, therefore c, G, etc., are always spelled out explicitly.

2. I assume the $(+,-,-,-)$ signature, i.e., the space-time interval is defined by $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ (same convention as in [1]).

3. Categories of space-time intervals:

$$\begin{aligned}
&\textbf{spacelike} : \text{if } ds^2 < 0 \quad \Rightarrow \quad dx^2 + dy^2 + dz^2 > c^2dt^2 \\
&\textbf{lightlike} : \text{if } ds^2 = 0 \quad \Rightarrow \quad dx^2 + dy^2 + dz^2 = c^2dt^2 \\
&\textbf{timelike} : \text{if } ds^2 > 0 \quad \Rightarrow \quad dx^2 + dy^2 + dz^2 < c^2dt^2
\end{aligned}$$

4. Latin letters indicate space variables in 3D space or in a generic n-dimensional space, while greek letters denote space-time variables in 4D space.

5. The previous items imply that the Minkowski metric is specified by the following metric tensor

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{1}$$

i.e., $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$.

6. Indexes follow the Einstein convention: repeated indexes imply a summation. For example, when we denote the inverse metric tensor with $\eta_{\mu\nu}$, and notice that

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2}$$

then we conclude that the matrix representation of $\eta_{\mu\nu}$ is the same as that of its inverse $\eta^{\mu\nu}$, and clearly $\eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^{\mu}_{\nu}$.

7. Lorentz transformations along a specific axis (x, in this case) are defined by

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \tag{3}$$

with $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$, as usual.

8. Proper time τ is time in the rest frame of the observer. Therefore, if something is at rest in the observer's frame (i.e., $dx = dy = dz = 0$), $ds^2 = -c^2 d\tau^2$.

1.1 Problem: express the space-time interval ds^2 of SR space-time in spherical coordinates.

Cartesian coordinates are expressed in terms of spherical coordinates as follows:

$$x = r \sin\theta \cos\varphi \tag{4a}$$
$$y = r \sin\theta \sin\varphi \tag{4b}$$
$$z = r \cos\theta \tag{4c}$$

Therefore

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \varphi} d\varphi = \sin\theta\cos\varphi dr + r\cos\theta\cos\varphi d\theta - r\sin\theta\sin\varphi d\varphi \tag{5a}$$
$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \varphi} d\varphi = \sin\theta\sin\varphi dr + r\cos\theta\sin\varphi d\theta + r\sin\theta\cos\varphi d\varphi \tag{5b}$$
$$dz = \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \theta} d\theta + \frac{\partial z}{\partial \varphi} d\varphi = \cos\theta dr - r\sin\theta d\theta \tag{5c}$$

Finally, squaring and summing, one finds:

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2. \tag{6}$$

2 Dynamics

To introduce dynamics, we must define first a 4-vector velocity. This is done as follows:

$$U^{\mu} = \frac{dx^{\mu}}{d\tau} = \begin{pmatrix} cdt/d\tau \\ dx/d\tau \\ dy/d\tau \\ dz/d\tau \end{pmatrix}. \tag{7}$$

Since

$$\frac{dx^0}{d\tau} = \frac{cdt}{d\tau} = c\gamma; \quad \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\,\frac{dt}{d\tau} = \gamma v^i \tag{8}$$

then

$$U^\mu = \begin{pmatrix} \gamma c \\ \gamma \mathbf{v} \end{pmatrix}. \tag{9}$$

Here we note in passing that the four-velocity has an invariant magnitude

$$\eta_{\mu\nu}U^\mu U^\nu = \gamma^2 c^2 - \gamma^2 v^2 = c^2\gamma^2\left(1 - \frac{v^2}{c^2}\right) = c^2. \tag{10}$$

4-momentum is defined as follows

$$P^\mu = mU^\mu = \begin{pmatrix} m\gamma c \\ \gamma m\mathbf{v} \end{pmatrix} = \begin{pmatrix} E/c \\ \mathbf{p} \end{pmatrix}, \tag{11}$$

therefore

$$\eta_{\mu\nu}P^\mu P^\nu = \frac{E^2}{c^2} - |\mathbf{p}|^2 = m^2c^2, \tag{12}$$

or also

$$E^2 = m^2c^4 + c^2|\mathbf{p}|^2. \tag{13}$$

Finally, the special relativistic extension of Newton's second law is

$$F^\mu = \frac{dP^\mu}{d\tau}. \tag{14}$$

2.1 Velocity transformations in the low–relative–velocity limit

Consider the following velocity transformation to a frame moving with relative speed β with respect to the original one in direction x:

$$v'_x = \frac{\Delta x'}{\Delta t'} = \frac{\gamma(\Delta x - \beta c\Delta t)}{\gamma(c\Delta t - \beta\Delta x)/c} = \frac{v_x - \beta c}{1 - \beta v_x/c} \underset{|\beta|\ll 1}{\longrightarrow} v_x - \beta c, \tag{15}$$

i.e., in the case of low relative velocity, the Lorentz transformation for velocity becomes the usual Galilean velocity transformation. Note also that the SR velocity transformation is nonlinear.

2.2 The Doppler effect

Consider two reference frames: one of the observer the other one of a light–emitting source: we take reference frames with parallel axes such that the source moves in direction x^1 with relative speed v and the beam of light lies in the (x^1, x^2) plane and is emitted by the source with an angle θ with respect to the x^1 axis. The wavelength of the emitted light is λ, and this means that the 4-vector that represents it in the source frame is

$$\frac{2\pi}{\lambda}\begin{pmatrix} 1 \\ \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \tag{16}$$

Therefore, applying the Lorentz transformation to transform this 4-vector to the observer frame we find

$$\frac{1}{\lambda_o} = \frac{1}{\lambda}\left(\gamma - \gamma\frac{v}{c}\cos\theta\right) \tag{17}$$

For a source that is approaching the observer, with $\theta = 0$, we find

$$\frac{\lambda}{\lambda_o} = \left(\gamma - \gamma\frac{v}{c}\right) = \frac{1}{\lambda}\left(\frac{1 - v/c}{1 + v/c}\right)^{1/2} \underset{|\mathrm{v}| \ll \mathrm{c}}{\longrightarrow} 1 - v/c \tag{18}$$

i.e., $\lambda < \lambda_o$ (blue shift). Similarly, for a source that is receding from the observer, we find

$$\frac{\lambda}{\lambda_o} = \left(\gamma - \gamma\frac{v}{c}\right) = \frac{1}{\lambda}\left(\frac{1 + v/c}{1 - v/c}\right)^{1/2} \underset{|\mathrm{v}| \ll \mathrm{c}}{\longrightarrow} 1 + v/c \tag{19}$$

i.e., $\lambda > \lambda_o$ (red shift).

If $\theta = \pm\pi/2$ we find a uniquely relativistic effect

$$\frac{\lambda}{\lambda_o} = \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2}v^2/c^2 \underset{|\mathrm{v}| \ll \mathrm{c}}{\longrightarrow} 1 \tag{20}$$

This *transverse Doppler effect* disappears at first order v/c in the non-relativistic case. It was first experimentally confirmed in 1938 by Ives and Stilwell [4]. Their experiment represents one of the essential tests of the validity of Special Relativity[1]. Previous attempts to confirm the transverse Doppler effect were unsuccessful, because they tried a direct measurement, with ion beams which had a low speed, $v \approx 0.005c$, and therefore produced a relative shift of the order of $1 + 2.5\ 10^{-5}$, too small to detect when taking into account all the sources of uncertainty.

Ives and Stilwell succeeded because they devised a differential measurement: they shone both a beam of light and its reflection on an ion beam. In this way they detected the both the redshifted (λ_R) and the blueshifted (λ_B) versions of the light, so that taking their averages, whatever the angle $\cos\theta$,

$$\frac{\lambda_B + \lambda_R}{2\lambda_o} = \gamma \tag{21}$$

i.e. the measurement is sensitive to the transverse Doppler effect alone, and with this arrangement they got rid of most of the sources of uncertainty.

[1]The webpage by J. Baez on the experimental basis of SR is a useful resource that lists the relevant experimental validations of SR, see `https://math.ucr.edu/home/baez/physics/Relativity/SR/experiments.html`

3 Example: the GZK cutoff

Cosmic rays are mainly protons and occasionally helium or heavier atomic nuclei and very occasionally electrons. They are often very energetic and some of them are the fastest particles with respect to Earth, moving at speeds very close to the speed of light. The *Oh-My-God particle* recorded over Utah in 1991 by the Fly's Eye detector was probably a proton traveling with an energy close to 3×10^{20} eV, many times higher than the highest energy provided by LHC[2].

Not all cosmic rays are so energetic, and their energies are distributed as in figure 1 (from `http://www.cosmic-ray.org/reading/flyseye.html`), which displays the cosmic–ray energy spectrum multiplied times E^2 to produce a better representation. Figure 2 shows yet another representation where the spectrum is multiplied times E^3 to display the changing spectral slope [7]:

- initially, at energies lower than about 10^{14} eV, the cosmic–ray energy spectrum behaves as a simple power law $\sim E^{-2.7}$;
- at higher energies, the spectrum exhibits significant structures which reflect the cosmic ray origins and propagation; above 10^{14} eV, the spectrum steepens with a break at 10^{14} eV known as the "knee";
- another "knee" near 3×10^{17} eV has been reported by several experiments;
- at still higher energies, we find an "ankle" structure followed by dip near $3 - 5 \times 10^{18}$ eV;
- all this is followed by final cutoff – the so-called *GZK cutoff.*

Understanding the cosmic–ray spectrum is an important scientific challenge. Most cosmic–ray particles with energies $< 10^{10}$ eV are known to originate from solar flares, others originate from elsewhere in the Galaxy. It is difficult to determine the sources of cosmic rays because magnetic fields in the Galaxy and in the Solar System distort their trajectories and their distribution, as seen from Earth, is roughly uniform in the sky.

Of all the structures in the cosmic–ray spectrum, the GZK cutoff is the easiest to understand. It was predicted in 1966 in two independent papers by K. Greisen and by G. Zatsepin and V. Kuz'min [2, 8]. The description given here follows those given in [3] (Box 5.1) and [6] (Box 3.5). Greisen,G. Zatsepin, and V. Kuz'min noted that sufficiently energetic photons can initiate a reaction with the photons of the cosmic microwave background (CMB)

$$p + \gamma \to N + \pi^{0/+} \tag{22}$$

[2] for a description of the Fly's Eye detector, see `http://www.cosmic-ray.org/reading/flyseye.html`

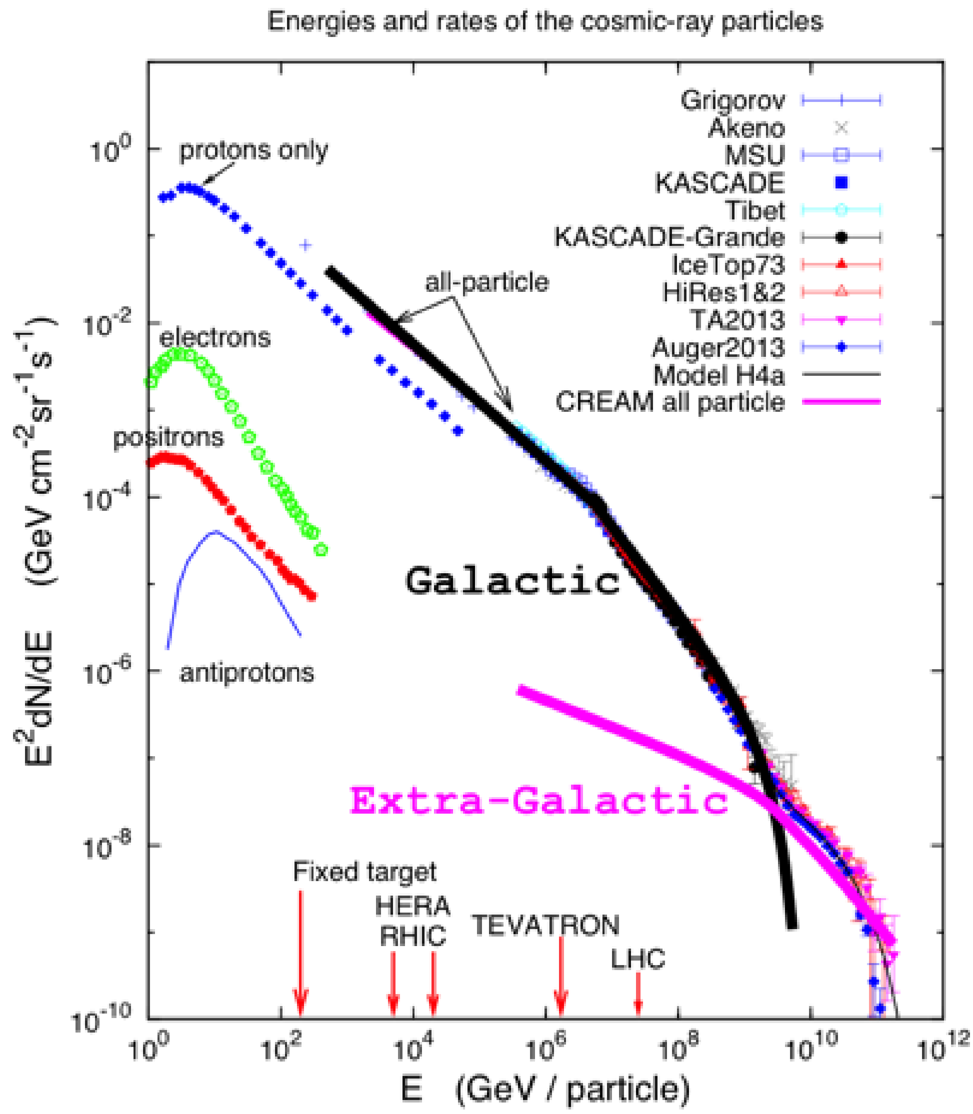

Figure 1: The cosmic–ray spectrum taken from `https://masterclass.icecube.wisc.edu/en/analyses/cosmic-ray-energy-spectrum`. Here, the spectrum is multiplied times E^2 to produce a better representation.

(N represents a nucleon, which means either a neutron or a proton; in the following we shall consider the neutral pion photoproduction, which implies a more stringent limit).

The photons in the CMB are distributed according to a black body spectrum with temperature 2.72548 ± 0.00057 K, corresponding to the modal photon energy of about

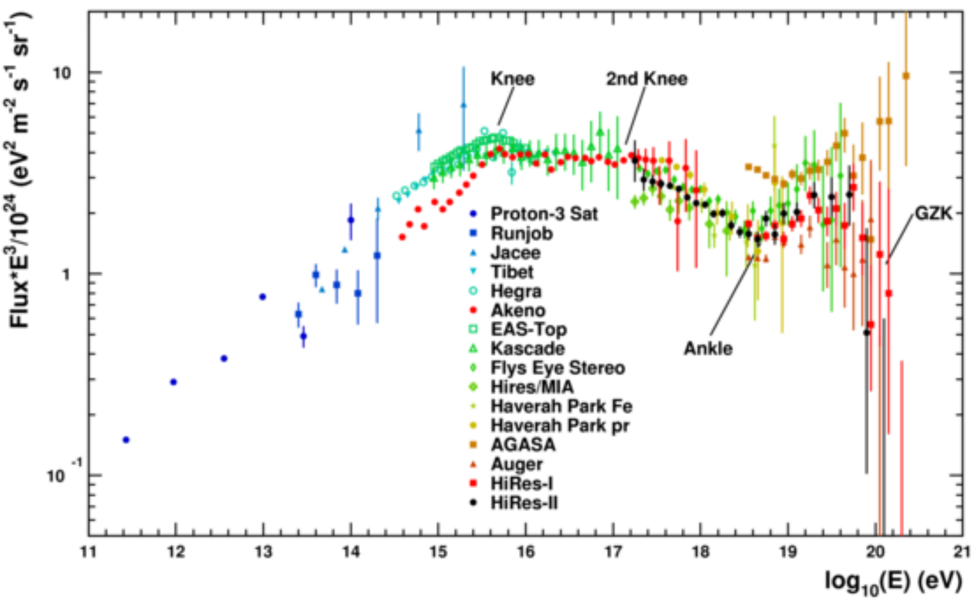

Figure 2: The cosmic–ray spectrum taken from [7]. Here, the spectrum is multiplied times E^3 to display the changing spectral slope, for a description see the main text.

6.626×10^{-4} eV (see Fig. 4). Given this photon energy, GZK calculated the proton energy at which the reaction (22) becomes important. Now, we denote with p_γ the photon 4-momentum, with p_p the proton 4-momentum, and with p_{π^0} the pion 4-momentum in the reaction

$$p + \gamma \to p + \pi^0,$$

then

$$p_\gamma + p_p = p'_p + p'_{\pi^0} \tag{23}$$

where the primes denote the final 4-momenta.

At the production threshold, the final 4-momenta have vanishing 3-momenta in the CM, so that in this reference frame,

$$p'_p + p'_{\pi^0} = \begin{pmatrix} m_p c + m_{\pi^0} c \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

therefore, we obtain the relativistic invariant

$$(p_\gamma + p_p)^2 = (p'_p + p'_{\pi^0})^2 = (m_p + m_{\pi^0})^2 c^2 \tag{24}$$

which holds in all frames of reference.

Figure 3: The Fly's Eye array operated out of Dugway Proving Ground, a military base in the desert of western Utah, from 1981 to 1993; it pioneered the "air fluorescence technique" for determining the energies and directions of ultrahigh-energy cosmic rays based on faint light emitted by nitrogen air molecules as the cosmic-ray air shower traverses the atmosphere. In 1991, the Fly's Eye detected a cosmic ray that still holds the world record for highest-energy particle (from `https://www.quantamagazine.org/the-particle-that-broke-a-cosmic-speed-limit-20150514`).

Now, recall that

$$p_\gamma^2 = 0; \quad p_p^2 = m_p c^2,$$

therefore we obtain

$$(p_\gamma + p_p)^2 = m_p^2 c^2 + 2 p_\gamma \cdot p_p = (m_p + m_{\pi^0})^2 c^2, \tag{25}$$

i.e.,

$$2 p_\gamma \cdot p_p = (m_p + m_{\pi^0})^2 c^2 - m_p^2 c^2 = m_{\pi^0}(2m_p + m_{\pi^0}) c^2 \tag{26}$$

Next, note that, at very high energy in the lab frame ($E_p \gg m_p c^2$),

$$E_p^2 - |\mathbf{p}_p|^2 c^2 = m_p^2 c^4 \quad \Rightarrow \quad E_p \approx |\mathbf{p}_p| c$$

and therefore, when we consider a head–to–head proton–photon collision along the x

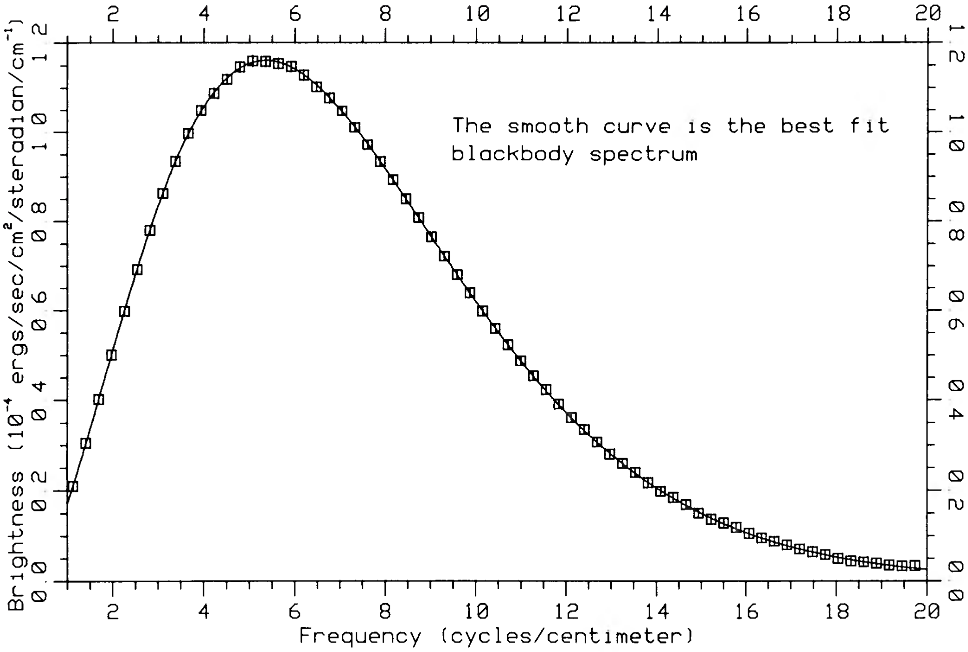

Figure 4: Preliminary spectrum of the cosmic microwave background from the FIRAS instrument on the Cosmic Background Explorer (COBE) mission at the north Galactic pole, compared to a blackbody. Boxes are measured points and show size of assumed 1% error band. The units for the vertical axis are 10^{-4} erg s^{-1} cm^{-2} sr^{-1} cm. Figure and caption taken from [5]. See also `https://science.nasa.gov/mission/cobe/`.

direction, the 4-momenta in the lab frame are

$$p_\gamma = \begin{pmatrix} E_\gamma/c \\ -E_\gamma/c \\ 0 \\ 0 \end{pmatrix}; \quad p_p = \begin{pmatrix} E_p/c \\ E_p/c \\ 0 \\ 0 \end{pmatrix}$$

so that

$$2p_\gamma \cdot p_p = 4\frac{E_p E_\gamma}{c^2} = m_{\pi^0}(2m_p + m_{\pi^0})c^2, \tag{27}$$

and finally,

$$E_p = \frac{m_{\pi^0}(2m_p + m_{\pi^0})c^4}{4E_\gamma} \tag{28}$$

is the **threshold proton energy** for π^0 production, for a given E_γ. When we take the modal CMB photon energy $E_\gamma \approx 6.626 \times 10^{-4}$ eV and the proton and charged pion masses $m_p c^2 \approx 938$ MeV and $m_{\pi^0} c^2 \approx 135$ MeV we find $E_p \approx 10^{20}$ eV.

Exercise:

1. using the well–know formulas for the density of black body photons at a given temperature T, find the CMB photon density;
2. at a proton energy $E_p \approx 10^{20}$ eV, the photoproduction cross section for the process

$$p + \gamma \to p + \pi^0,$$

is about 2×10^{-28} cm^2, and nearly energy–independent ; using this information and the density of CMB photons, find the corresponding mean free path of protons.

References

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