Einstein's equations

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By the principle of general covariance we know that the basic equations of GR should have the generic form

$$\text{some tensor of rank } (r, s) = \text{another tensor of rank } (r, s)$$

where the tensor on the l.h.s. should contain information about the curvature of spacetime, and the tensor on the r.h.s. should be the source term. More specifically, we can guess that on the l.h.s. we expect to find something related to the Riemann tensor, and on the r.h.s. something that is proportional to the stress-energy tensor.

When Einstein set out to find the exact structure of the equations, he tried many options, mostly guided by the principle of consistency, which calls for equations that transform into the usual Newtonian expressions for suitably chosen conditions, and also by some important experimental facts, like the measured precession of Mercury's perihelion.

This was no easy task. Since the Riemann tensor is a rank-4 tensor while the stress-energy tensor is a rank-2 tensor, Einstein tried a straightforward option when he took the Ricci tensor as a "condensed" version of the Riemann tensor

$$R^{\mu\nu} = \kappa T^{\mu\nu} \tag{1}$$

Such an equation correctly predicts the precession of Mercury's perihelion, BUT it does not satisfy the local conservation of energy

$$\nabla_\mu R^{\mu\nu} \neq 0 \quad \text{while} \quad \nabla_\mu T^{\mu\nu} = 0 \tag{2}$$

Indeed, we can show that the covariant derivative in general does not vanish by taking the Bianchi identity

$$\nabla_\sigma R_{\mu\alpha\beta\gamma} + \nabla_\beta R_{\mu\alpha\gamma\sigma} + \nabla_\gamma R_{\mu\alpha\sigma\beta} = 0 \tag{3}$$

and contracting it twice

$$g^{\mu\sigma} g^{\alpha\gamma} \left(\nabla_\sigma R_{\mu\alpha\beta\gamma} + \nabla_\beta R_{\mu\alpha\gamma\sigma} + \nabla_\gamma R_{\mu\alpha\sigma\beta}\right) = 0 \tag{4}$$

and finally demonstrate that in general $\nabla_\mu R^{\mu\nu} \neq 0$. Examining each term in (4), using the internal symmetries of the Riemann tensor and recalling that the covariant derivative of the metric tensor vanishes, we find

$$1.\quad g^{\mu\sigma} g^{\alpha\gamma} \nabla_\sigma R_{\mu\alpha\beta\gamma} = \nabla^\mu g^{\alpha\gamma} R_{\mu\alpha\beta\gamma} = \nabla^\mu g^{\alpha\gamma} R_{\alpha\mu\gamma\beta} = \nabla^\mu R^\gamma_{\mu\gamma\beta} = \nabla^\mu R_{\mu\beta} \tag{5}$$

$$2.\quad g^{\mu\sigma} g^{\alpha\gamma} \nabla_\beta R_{\mu\alpha\gamma\sigma} = \nabla_\beta g^{\mu\sigma} g^{\alpha\gamma} R_{\mu\alpha\gamma\sigma} = \nabla_\beta g^{\alpha\gamma} R^\sigma_{\alpha\gamma\sigma} = -\nabla_\beta g^{\alpha\gamma} R^\sigma_{\alpha\sigma\gamma}$$

$$= -\nabla_\beta g^{\alpha\gamma} R_{\alpha\gamma} = -\nabla_\beta R \tag{6}$$

$$3.\quad g^{\mu\sigma} g^{\alpha\gamma} \nabla_\gamma R_{\mu\alpha\sigma\beta} = \nabla^\alpha g^{\mu\sigma} R_{\mu\alpha\sigma\beta} = \nabla^\alpha R^\sigma_{\alpha\sigma\beta} = \nabla^\alpha R_{\alpha\beta} \tag{7}$$

Therefore

$$\nabla^\alpha R_{\alpha\beta} = \frac{1}{2} \nabla_\beta R \tag{8}$$

which, in general manifolds where the curvature is not constant, does not vanish.

Overall, it was a long and bumpy road for Einstein (for a historical account see [2]; see also [1] that settles the dispute on the priority of the discovery of the equations), but he finally made it. The key is again (8) which we can write in the modified form

$$\nabla^\alpha R_{\alpha\beta} = \frac{1}{2} \nabla_\beta R = \frac{1}{2} \nabla^\alpha g_{\alpha\beta} R \tag{9}$$

which leads to

$$\nabla^\alpha \left(R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R \right) = 0 \tag{10}$$

i.e., the **Einstein tensor**

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \tag{11}$$

has a vanishing covariant derivative and can be used to set up **Einstein's equations**

$$G^{\mu\nu} = \kappa T^{\mu\nu}. \tag{12}$$

Einstein's equations

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \kappa T^{\mu\nu}. \tag{13}$$

determine the independent components of the metric tensor, however not all 10 of them, since $G^{\mu\nu}$ satisfies the 4 equations $\nabla_\mu G^{\mu\nu} = 0$. This reflects the fact that we are free to transform to a different coordinate system with 4 coordinate transformation equations: $x'^\mu = f^\mu(\mathbf{x})$. Finally, this means that an equivalent of 4 Einstein's equation is automatically satisfied, so that we only have 6 independent degrees of freedom in $G^{\mu\nu}$.

Finally, we can easily find an alternate form of Einstein's equations by first contracting the equations

$$g_{\mu\nu} R^{\mu\nu} - \frac{1}{2} g_{\mu\nu} g^{\mu\nu} R = \kappa g_{\mu\nu} T^{\mu\nu} \quad \Rightarrow \quad R - \frac{1}{2} \delta^\mu_\mu R = \kappa T \quad \Rightarrow -R = \kappa T \tag{14}$$

where $\delta^\mu_\mu = 4$ and $T = g_{\mu\nu}T^{\mu\nu}$. Therefore we find

$$R^{\mu\nu} + \frac{\kappa}{2} g^{\mu\nu} T = \kappa T^{\mu\nu} \tag{15}$$

and finally

$$R^{\mu\nu} = \kappa \left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T \right) \tag{16}$$

As a final remark, I add here that any field proportional to the metric tensor $\Lambda g_{\mu\nu}$ – with Λ a proportionality constant called the **cosmological constant** – satisfies the condition $\nabla_\mu(\Lambda g_{\mu\nu}) = 0$ and can be added to the Einstein tensor, so that

$$G^{\mu\nu} + \Lambda g^{\mu\nu} = \kappa T^{\mu\nu} \tag{17}$$

also holds. This is also equivalent to

$$G^{\mu\nu} = \kappa T^{\mu\nu} - \Lambda g^{\mu\nu} = \kappa \left(T^{\mu\nu} - \frac{\Lambda}{\kappa} g^{\mu\nu} \right) \tag{18}$$

and to a vacuum energy density

$$\rho_{\text{vacuum}} c^2 = -\frac{\Lambda}{\kappa} \tag{19}$$

References

[1] Leo Corry, Jurgen Renn, and John Stachel. Belated decision in the Hilbert-Einstein priority dispute. *Science*, 278(5341):1270–1273, 1997.

[2] John Earman and Clark Glymour. Lost in the tensors: Einstein's struggles with covariance principles 1912–1916. *Studies In History and Philosophy of Science Part A*, 9(4):251–278, 1978.