The Newtonian limit

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1 Equations of motion in a gravitational potential

The Newtonian limit of General Relativity is defined by the conditions

- all motions are slow ($v \ll c$)
- gravitational fields are weak
- gravitational fields are static

From these conditions we can retrieve the usual equation of motion in a gravitational field

$$\frac{d^2 x^i}{dt^2} = -\partial_i \Phi,$$

where Φ is the nonrelativistic gravitational potential. To this end, we note first that by the assumption that all motions are slow

$$\frac{dx^i}{d\tau} \ll c \quad \Rightarrow \quad \frac{dx^i}{d(c\tau)} \ll 1 \quad \text{and} \quad \frac{dt}{d\tau} \approx 1$$

then the only non-negligible terms in the geodesic equations are those with the Γ^α_{00} Christoffel symbols

$$\frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0 \quad \Rightarrow \quad \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{00} \left(\frac{cdt}{d\tau} \right)^2 \approx 0, \tag{1}$$

Using the expression

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\nu} \left(\frac{\partial g_{\nu\gamma}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\nu} \right), \tag{2}$$

and the assumption of static field, so that the time derivatives vanish, we find

$$\Gamma^\alpha_{00} = \frac{1}{2} g^{\alpha\nu} \left(-\frac{\partial g_{00}}{\partial x^\nu} \right) = -\frac{1}{2} g^{\alpha\nu} \frac{\partial g_{00}}{\partial x^\nu}, \tag{3}$$

Finally, from the assumption that fields are weak, we consider the metric tensor to be approximately equal to $\eta_{\mu\nu}$ but for a small perturbation $h_{\mu\nu}$ (such that $|h_{\mu\nu}| \ll 1$):

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}. \tag{4}$$

The inverse metric tensor has a similar expression

$$g^{\mu\nu} \approx \eta^{\mu\nu} + h'^{\mu\nu}, \tag{5}$$

and by contracting it with the metric tensor we obtain

$$\delta^{\mu}_{\nu} = g_{\mu\alpha} g^{\alpha\nu} \approx (\eta_{\mu\alpha} + h_{\mu\alpha})(\eta^{\alpha\nu} + h'^{\alpha\nu}) \approx \delta^{\mu}_{\nu} + \eta_{\mu\alpha} h'^{\alpha\nu} + h_{\mu\alpha}\eta^{\alpha\nu}, \tag{6}$$

therefore

$$\eta_{\mu\alpha} h'^{\alpha\nu} = -h_{\mu\alpha}\eta^{\alpha\nu}, \tag{7}$$

i.e.,

$$h'^{\mu\nu} = -\eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta}. \tag{8}$$

Accordingly, from

$$h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta} = -h'^{\mu\nu}, \tag{9}$$

we obtain the expression for the inverse metric tensor

$$g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}. \tag{10}$$

Going back to eq. (3), we find that these definitions imply

$$\Gamma^{\alpha}_{00} \approx -\frac{1}{2}\eta^{\alpha\nu}\frac{\partial h_{00}}{\partial x^{\nu}}, \tag{11}$$

and the geodesic equation

$$\frac{d^2 x^{\alpha}}{d\tau^2} + \Gamma^{\alpha}_{00} c^2 = \frac{d^2 x^{\alpha}}{d\tau^2} - \frac{c^2}{2}\eta^{\alpha\nu}\frac{\partial h_{00}}{\partial x^{\nu}} \approx 0, \tag{12}$$

The field is static, therefore

$$\frac{d^2 x^0}{d\tau^2} - \frac{c^2}{2}\eta^{0\nu}\frac{\partial h_{00}}{\partial x^{\nu}} = \frac{d^2 x^0}{d\tau^2} - \frac{c^2}{2}\frac{\partial h_{00}}{\partial x^0} \approx 0, \tag{13}$$

i.e.,

$$\frac{d^2 t}{d\tau^2} \approx \frac{c}{2}\frac{\partial h_{00}}{\partial t} = 0, \tag{14}$$

so that $\frac{dt}{d\tau}$ is constant.

Moving now to the space coordinates, we find

$$\frac{d^2 x^i}{d\tau^2} - \frac{c^2}{2}\eta^{i\nu}\frac{\partial h_{00}}{\partial x^{\nu}} = \frac{d^2 x^i}{d\tau^2} + \frac{c^2}{2}\frac{\partial h_{00}}{\partial x^i} \approx 0, \tag{15}$$

Finally, setting $h_{00} = 2\Phi/c^2$ we obtain

$$\frac{d^2 x^i}{d\tau^2} \approx -\frac{\partial \Phi}{\partial x^i}, \tag{16}$$

which is the usual equation of motion with gravitational potential Φ.

2 Poisson's equation

Here, we use the classical Poisson's equation for the gravitational field

$$\nabla^2\Phi = 4\pi G\rho, \tag{17}$$

where ρ is the mass density, to determine the value of the Einstein's constant κ. To this end we assume once again that the metric tensor is approximately equal to the Minkowski tensor with the addition of a small perturbation $|h_{\mu\nu}| \ll 1$):

$$g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}. \tag{18}$$

Using the alternate form of Einstein's field equations

$$R^{\mu\nu} = \kappa\left(T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T\right), \tag{19}$$

with $T^{\mu\nu} = \rho U^\mu U^\nu$, we find

$$R^{\mu\nu} \approx \kappa\left(\rho U^\mu U^\nu - \frac{1}{2}(\eta^{\mu\nu} - h^{\mu\nu})\rho c^2\right). \tag{20}$$

Since the only non-zero component of the stress-energy tensor is T^{00}, we focus on the case $\mu = \nu = 0$. Moreover, at low speed $U^\mu \approx (c, \mathbf{v})$, therefore

$$R^{00} \approx \kappa\left(\rho c^2 - \frac{1}{2}\rho c^2\right) = \frac{1}{2}\rho c^2\kappa \tag{21}$$

From the explicit expression of the Ricci tensor

$$R_{\mu\nu} = \partial_\alpha\Gamma^\alpha_{\mu\nu} - \partial_\mu\Gamma^\alpha_{\alpha\nu} + \Gamma^\alpha_{\alpha\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\alpha\nu}, \tag{22}$$

we obtain

$$R_{00} = \partial_\alpha\Gamma^\alpha_{00} - \partial_0\Gamma^\alpha_{\alpha 0} + \Gamma^\alpha_{\alpha\sigma}\Gamma^\sigma_{00} - \Gamma^\alpha_{0\sigma}\Gamma^\sigma_{\alpha 0} \approx \partial_\alpha\Gamma^\alpha_{00} - \partial_0\Gamma^\alpha_{\alpha 0}, \tag{23}$$

where the final approximation comes from the consideration that the connection coefficients must be small in this case and we get rid of second-order terms. Moreover, we assume a static field, therefore the time derivative vanishes and we find

$$R_{00} \approx \partial_\alpha\Gamma^\alpha_{00}. \tag{24}$$

We already found that in these conditions

$$\Gamma^\alpha_{00} \approx -\frac{1}{2}\eta^{\alpha\nu}\frac{\partial h_{00}}{\partial x^\nu}, \tag{25}$$

therefore

$$R_{00} \approx -\frac{1}{2}\eta^{\alpha\nu}\partial_\alpha\partial_\nu h_{00} = -\frac{1}{2}\partial_\alpha\partial^\alpha h_{00} = -\frac{1}{2}\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)h_{00}. \tag{26}$$

From (21) and (26) we obtain the equality

$$\frac{1}{2}\rho c^2\kappa = \frac{1}{2}\nabla^2 h_{00}, \tag{27}$$

and combining this result with $h_{00} = 2\Phi/c^2$, we finally find

$$\frac{1}{2}\rho c^2\kappa = \frac{1}{c^2}\nabla^2\Phi, \tag{28}$$

i.e.,

$$\nabla^2\Phi = \frac{1}{2}\rho c^4\kappa, \tag{29}$$

and comparing this with Poisson's equation, we find

$$\kappa = \frac{8\pi G}{c^4}, \tag{30}$$

so that Einstein's equations are

$$G^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu}. \tag{31}$$