Linearized gravity

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1 Linearized gravity

When dealing with the Newtonian limit of General Relativity we have already met the weak-field condition, whereby the metric tensor is approximately equal to $\eta_{\mu\nu}$ but for a small perturbation $h_{\mu\nu}$ (such that $|h_{\mu\nu}| \ll 1$)

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

and the inverse metric tensor is

$$g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}. \tag{2}$$

These definitions imply that the connection coefficients can be approximated as follows,

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta} \right) \tag{3}$$

$$\approx \frac{1}{2} \eta^{\mu\nu} \left(\partial_\alpha h_{\nu\beta} + \partial_\beta h_{\nu\alpha} - \partial_\nu h_{\alpha\beta} \right). \tag{4}$$

The same goes for the Riemann tensor

$$R^{\mu}_{\alpha\beta\gamma} = \partial_\beta \Gamma^{\mu}_{\alpha\gamma} - \partial_\gamma \Gamma^{\mu}_{\alpha\beta} + \Gamma^{\delta}_{\alpha\gamma}\Gamma^{\mu}_{\delta\beta} - \Gamma^{\delta}_{\alpha\beta}\Gamma^{\mu}_{\delta\gamma} \approx \partial_\beta \Gamma^{\mu}_{\alpha\gamma} - \partial_\gamma \Gamma^{\mu}_{\alpha\beta} \tag{5}$$

$$\approx \frac{1}{2} \eta^{\mu\nu} \left[\partial_\beta \left(\partial_\alpha h_{\nu\gamma} + \partial_\gamma h_{\nu\alpha} - \partial_\nu h_{\alpha\gamma} \right) - \partial_\gamma \left(\partial_\alpha h_{\nu\beta} + \partial_\beta h_{\nu\alpha} - \partial_\nu h_{\alpha\beta} \right) \right] \tag{6}$$

$$\approx \frac{1}{2} \eta^{\mu\nu} \left[\left(\partial_\alpha \partial_\beta h_{\nu\gamma} + \partial_\beta \partial_\gamma h_{\nu\alpha} - \partial_\beta \partial_\nu h_{\alpha\gamma} \right) - \left(\partial_\alpha \partial_\gamma h_{\nu\beta} + \partial_\beta \partial_\gamma h_{\nu\alpha} - \partial_\nu \partial_\gamma h_{\alpha\beta} \right) \right] \tag{7}$$

$$= \frac{1}{2} \eta^{\mu\nu} \left(\partial_\alpha \partial_\beta h_{\nu\gamma} - \partial_\beta \partial_\nu h_{\alpha\gamma} - \partial_\alpha \partial_\gamma h_{\nu\beta} + \partial_\nu \partial_\gamma h_{\alpha\beta} \right) \tag{8}$$

Correspondingly, we find a linearized expression for the Ricci tensor

$$R_{\alpha\beta} = R^{\mu}_{\alpha\beta\mu} \approx \frac{1}{2} \eta^{\mu\nu} \left(\partial_\alpha \partial_\beta h_{\nu\mu} - \partial_\beta \partial_\nu h_{\alpha\mu} - \partial_\alpha \partial_\mu h_{\nu\beta} + \partial_\nu \partial_\mu h_{\alpha\beta} \right) \tag{9}$$

$$= \frac{1}{2} \left(\partial_\alpha \partial_\beta h^{\mu}_{\mu} - \partial_\beta \partial^\mu h_{\alpha\mu} - \partial_\alpha \partial^\mu h_{\mu\beta} + \partial^\mu \partial_\mu h_{\alpha\beta} \right) \tag{10}$$

$$= \frac{1}{2} \left(\Box^2 h_{\alpha\beta} + \partial_\alpha \partial_\beta h - \partial_\beta \partial^\mu h_{\alpha\mu} - \partial_\alpha \partial^\mu h_{\mu\beta} \right) \tag{11}$$

where $h = h^\mu_\mu$ and where

$$\Box^2 = \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \tag{12}$$

is the d'Alambertian operator. From this it follows that the Ricci scalar is

$$R = \eta^{\alpha\beta} R_{\alpha\beta} = \Box^2 h - \partial_\mu \partial_\nu h^{\mu\nu}, \tag{13}$$

and finally we obtain the Einstein's tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R = \frac{1}{2}\left(\Box^2 h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\mu \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha h_{\mu\alpha} - \eta_{\mu\nu}\Box^2 h + \eta_{\mu\nu}\partial_\alpha \partial_\beta h^{\alpha\beta}\right), \tag{14}$$

and the Einstein equations for linearized gravity

$$G_{\mu\nu} = \frac{1}{2}\left(\Box^2 h_{\mu\nu} + \partial_\mu \partial_\nu h - \partial_\mu \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha h_{\mu\alpha} - \eta_{\mu\nu}\Box^2 h + \eta_{\mu\nu}\partial_\alpha \partial_\beta h^{\alpha\beta}\right) = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{15}$$

Eq. (15) becomes slightly simpler by redefining the **trace-reversed perturbation variables**:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \tag{16}$$

so that

$$\bar{h} = \bar{h}^\mu_\mu = h - \frac{1}{2}\eta^{\mu\nu}\eta_{\mu\nu} h = h - \frac{1}{2}\delta^\mu_\mu h = h - 2h = -h \tag{17}$$

and therefore

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}. \tag{18}$$

Using the trace-reversed perturbation variables in eq (15), we find

$$\begin{aligned} &\Box^2 \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box^2 \bar{h} - \partial_\mu \partial_\nu \bar{h} - \partial_\mu \partial^\alpha \bar{h}_{\alpha\nu} + \frac{1}{2}\partial_\mu \partial_\nu \bar{h} \\ &\quad - \partial_\nu \partial^\alpha \bar{h}_{\mu\alpha} + \frac{1}{2}\partial_\nu \partial_\mu \bar{h} + \eta_{\mu\nu}\Box^2 \bar{h} + \eta_{\mu\nu}\partial_\alpha \partial_\beta \bar{h}^{\alpha\beta} - \frac{1}{2}\eta_{\mu\nu}\Box^2 \bar{h} = \frac{16\pi G}{c^4} T_{\mu\nu}, \end{aligned} \tag{19}$$

i.e.,

$$\Box^2 \bar{h}_{\mu\nu} - \partial_\mu \partial^\alpha \bar{h}_{\alpha\nu} - \partial_\nu \partial^\alpha \bar{h}_{\mu\alpha} + \eta_{\mu\nu}\partial_\alpha \partial_\beta \bar{h}^{\alpha\beta} = \frac{16\pi G}{c^4} T_{\mu\nu}, \tag{20}$$

which is the Einstein equation for linearized gravity with trace-reversed perturbation variables.