The transverse-traceless gauge

Edoardo Milotti

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1 Gauge transformations

We have already met the multiple degrees of freedom associated with the freedom of choice of the coordinate system, and here we note that small changes of the metric tensor can be due to:

1. perturbations of space-time
2. small transformations of the coordinate system
3. both 1. and 2.

We can understand the effect of small coordinate transformations by performing a **gauge transformation**, where we consider two coordinate systems which differ by a small translation ξ^μ:

$$x'^\mu = x^\mu + \xi^\mu; \quad x^\mu = x'^\mu - \xi^\mu \quad (|\xi^\mu| \ll 1) \tag{1}$$

so that the coordinate transformation matrices are

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu \xi^\mu \tag{2}$$

$$\frac{\partial x^\mu}{\partial x'^\nu} = \delta^\mu_\nu - \partial'_\nu \xi^\mu = \delta^\mu_\nu - \frac{\partial x^\alpha}{\partial x'^\nu}\frac{\partial \xi^\mu}{\partial x^\alpha} = \delta^\mu_\nu - (\delta^\alpha_\nu - \partial'_\nu \xi^\alpha)\frac{\partial \xi^\mu}{\partial x^\alpha} \approx \delta^\mu_\nu - \partial_\nu \xi^\mu, \tag{3}$$

where we have kept terms at most linear in ξ^μ and its derivatives.

In particular, the metric tensor (and similar rank-2 covariant tensors) transforms as follows:

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta} = (\delta^\alpha_\mu - \partial_\mu \xi^\alpha)(\delta^\beta_\nu - \partial_\nu \xi^\beta) g_{\alpha\beta} \tag{4}$$

$$\approx g_{\mu\nu} - \partial_\nu \xi^\beta \, g_{\mu\beta} - \partial_\mu \xi^\alpha \, g_{\alpha\nu} \tag{5}$$

$$= g_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu \tag{6}$$

Here, we make once again the assumptions at the basis of linearized gravity

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}, \tag{7}$$

and

$$g'_{\mu\nu} \approx \eta_{\mu\nu} + h'_{\mu\nu}, \tag{8}$$

therefore, when expressed in terms of perturbation variables, the transformation eq. (6) becomes

$$\eta_{\mu\nu} + h'_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu, \tag{9}$$

i.e.,

$$h'_{\mu\nu} \approx h_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu. \tag{10}$$

We find the transformation of the trace-reversed perturbation variables evaluating first the trace of the transformed variable

$$h' = \eta^{\mu\nu} h'_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu} - \eta^{\mu\nu} \partial_\nu \xi_\mu - \eta^{\mu\nu} \partial_\mu \xi_\nu = h - 2\partial_\mu \xi^\mu, \tag{11}$$

so that the trace-reversed perturbation variables become

$$\bar{h}'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h' = h_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu - \frac{1}{2}\eta_{\mu\nu} \left(h - 2\partial_\alpha \xi^\alpha\right) \tag{12}$$

$$= \bar{h}_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu + \eta_{\mu\nu} \partial_\alpha \xi^\alpha. \tag{13}$$

As they should, these coordinate transformations do not affect (at this order) the Riemann tensor:

$$R'^{\mu}_{\alpha\beta\gamma} = \frac{1}{2}\eta^{\mu\nu} \left(\partial_\alpha \partial_\beta h'_{\nu\gamma} - \partial_\beta \partial_\nu h'_{\alpha\gamma} - \partial_\alpha \partial_\gamma h'_{\nu\beta} + \partial_\nu \partial_\gamma h'_{\alpha\beta}\right) \tag{14}$$

$$= \frac{1}{2}\eta^{\mu\nu} \left[\partial_\alpha \partial_\beta (h_{\nu\gamma} - \partial_\nu \xi_\gamma - \partial_\gamma \xi_\nu) - \partial_\beta \partial_\nu (h_{\alpha\gamma} - \partial_\alpha \xi_\gamma - \partial_\gamma \xi_\alpha) \right.$$
$$\left. - \partial_\alpha \partial_\gamma (h_{\nu\beta} - \partial_\nu \xi_\beta - \partial_\beta \xi_\nu) + \partial_\nu \partial_\gamma (h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha)\right] \tag{15}$$

$$= \frac{1}{2}\eta^{\mu\nu} \left(\partial_\alpha \partial_\beta h_{\nu\gamma} - \partial_\beta \partial_\nu h_{\alpha\gamma} - \partial_\alpha \partial_\gamma h_{\nu\beta} + \partial_\nu \partial_\gamma h_{\alpha\beta}\right) = R^{\mu}_{\alpha\beta\gamma} \tag{16}$$

We use the available degrees of freedom to choose a suitable coordinate frame and simplify the Einstein equation for linearized gravity with trace-reversed perturbation variables

$$\Box^2 \bar{h}_{\mu\nu} - \partial_\mu \partial^\alpha \bar{h}_{\alpha\nu} - \partial_\nu \partial^\alpha \bar{h}_{\mu\alpha} + \eta_{\mu\nu} \partial_\alpha \partial_\beta \bar{h}^{\alpha\beta} = \frac{16\pi G}{c^4} T_{\mu\nu}. \tag{17}$$

We see that we can fix the coordinate system in a particularly advantageous way if it is possible to set

$$\partial_\nu \bar{h}^{\mu\nu} = 0 \tag{18}$$

(the **Lorentz gauge**), then it is straightforward to see that the equation reduces to the system

$$\Box^2 \bar{h}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu} \tag{19}$$

$$\partial_\nu \bar{h}^{\mu\nu} = 0 \tag{20}$$

1.1 Existence of the Lorentz gauge

Is there a coordinate system that actually satisfies the Lorentz condition? If it exists, then there must be a coordinate transformation

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu + \eta_{\mu\nu} \partial_\alpha \xi^\alpha \tag{21}$$

such that $\partial_\nu \bar{h}'^{\mu\nu} = 0$. Indeed, from eq. (21), we find

$$\begin{aligned} 0 = \partial^\nu \bar{h}'_{\mu\nu} &= \partial^\nu \bar{h}_{\mu\nu} - \partial^\nu \partial_\nu \xi_\mu - \partial^\nu \partial_\mu \xi_\nu + \eta_{\mu\nu} \partial^\nu \partial_\alpha \xi^\alpha & (22) \\ &= \partial^\nu \bar{h}_{\mu\nu} - \Box^2 \xi_\mu - \partial_\mu \partial^\nu \xi_\nu + \partial_\mu \partial^\nu \xi_\nu & (23) \\ &= \partial^\nu \bar{h}_{\mu\nu} - \Box^2 \xi_\mu & (24) \end{aligned}$$

We end up with the equation

$$\partial^\nu \bar{h}_{\mu\nu} = \Box^2 \xi_\mu \tag{25}$$

which is a wave equation with a source term. Linear differential equations such as this can always be solved by functions of the form $g(x) + g_0(x)$, where g is a particular solution which takes into account the source term on the r.h.s. of the equation, and g_0 is the general solution of the associated homogeneous equation

$$\partial^\nu \bar{h}_{\mu\nu} = 0, \tag{26}$$

therefore we conclude that we can *always* find a suitable transformation that takes us to a coordinate system where the Lorentz gauge holds.

2 The transverse-traceless gauge

In empty space the wave equation with the Lorentz condition

$$\begin{aligned} \Box^2 \bar{h}'_{\mu\nu} &= \frac{16\pi G}{c^4} T_{\mu\nu} & (27) \\ \partial_\nu \bar{h}'^{\mu\nu} &= 0 & (28) \end{aligned}$$

reduces to

$$\begin{aligned} \Box^2 \bar{h}'_{\mu\nu} &= 0 & (29) \\ \partial_\nu \bar{h}'^{\mu\nu} &= 0 & (30) \end{aligned}$$

and this works for a gauge transformation 4-vector ξ_μ that leads to a coordinate system that satisfies the Lorentz condition

$$\partial^\nu \bar{h}_{\mu\nu} = \Box^2 \xi_\mu \tag{31}$$

However with the choice of the Lorentz gauge we have not exhausted the degrees of freedom available to us. The $\bar{h}_{\mu\nu}$ symmetric tensor has 10 independent components and Eq. (31) corresponds to 4 scalar equations. This means that we have 6 degrees of freedom left and that there is an infinity of ξ_μ that satisfy this equation. Therefore, we can assume the Lorentz gauge and still have residual freedom to further constrain the coordinate system.

Now, we drop the prime and spell out the wave equation

$$\Box^2 \bar{h}_{\mu\nu} = \frac{1}{c^2}\frac{\partial^2 \bar{h}_{\mu\nu}}{\partial t^2} - \nabla^2 \bar{h}_{\mu\nu} = 0 \tag{32}$$

and look for solutions

$$\bar{h}^{\mu\nu} = \mathrm{Re}[A^{\mu\nu}\exp(ik_\alpha x^\alpha)] \tag{33}$$

Note that eq. (32) predicts the existence of waves that propagate at speed c. This is an important prediction of GR.

These trial solutions must satisfy the following constraints:

- from the symmetry of the metric tensor: $A^{\mu\nu} = A^{\nu\mu}$
- from the wave equation: $k^\alpha k_\alpha = 0$, i.e., the wave 4-vector is null; this condition implies the usual dispersion relation, i.e., waves move with speed c
- from the Lorentz condition: $k_\nu A^{\mu\nu} = 0$

Next, we consider a wave propagating in the x^3 direction, so that the wave 4-vector is

$$k^\mu = (k, 0, 0, k); \quad k_\mu = (k, 0, 0, -k) \tag{34}$$

(the equality $k^0 = k^3$ follows from the fact that this is a null vector), with $k = \omega/c$. With this choice of k^μ, we find the following equality from the Lorentz condition

$$kA^{\mu 0} - kA^{\mu 3} = 0, \tag{35}$$

i.e.,

$$A^{\mu 0} = A^{\mu 3}, \tag{36}$$

and the A matrix writes

$$[A^{\mu\nu}] = \begin{pmatrix} A^{00} & A^{01} & A^{02} & A^{00} \\ A^{01} & A^{11} & A^{12} & A^{01} \\ A^{02} & A^{12} & A^{22} & A^{02} \\ A^{00} & A^{01} & A^{02} & A^{00} \end{pmatrix} \tag{37}$$

We see that the A matrix depends on just 6 quantities, $(A^{00}, A^{01}, A^{02}, A^{11}, A^{12}, A^{22})$, in line with what we expect from the application of the Lorentz condition (the symmetric tensor has 10 independent components, but the Lorentz condition implies 4 equalities, and hence the number of independent components is reduced to 6).

Since the Lorentz gauge determines a whole class of gauge transformations, we need to further specify the coordinate transformation. We do this by choosing

$$\xi^\mu = -\mathrm{Re}[i\epsilon^\mu \exp(ik_\alpha x^\alpha)], \tag{38}$$

where k^μ is the same as in eq. (33), therefore **this is a wave-dependent gauge transformation**. This trivially satisfies the Lorentz condition, and moreover from

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu + \eta_{\mu\nu}\partial_\alpha \xi^\alpha \tag{39}$$

we find

$$A'^{\mu\nu} = A^{\mu\nu} - k^\nu \epsilon^\mu - \epsilon^\nu k^\mu + \eta^{\mu\nu} k_\alpha \epsilon^\alpha. \tag{40}$$

From eq. (40) and from the values of k^μ and $A^{\mu\nu}$ listed above, we find

$$A'^{00} = A^{00} - k^0\epsilon^0 - \epsilon^0 k^0 + \eta^{00} k_\alpha \epsilon^\alpha = A^{00} - k(\epsilon^0 + \epsilon^3) \tag{41}$$

$$A'^{01} = A^{01} - k^1\epsilon^0 - \epsilon^1 k^0 + \eta^{01} k_\alpha \epsilon^\alpha = A^{01} - k\epsilon^1 \tag{42}$$

$$A'^{02} = A^{02} - k^2\epsilon^0 - \epsilon^2 k^0 + \eta^{02} k_\alpha \epsilon^\alpha = A^{02} - k\epsilon^2 \tag{43}$$

$$A'^{11} = A^{11} - k^1\epsilon^1 - \epsilon^1 k^1 + \eta^{11} k_\alpha \epsilon^\alpha = A^{11} - k(\epsilon^0 - \epsilon^3) \tag{44}$$

$$A'^{12} = A^{12} - k^2\epsilon^1 - \epsilon^2 k^1 + \eta^{12} k_\alpha \epsilon^\alpha = A^{12} \tag{45}$$

$$A'^{22} = A^{22} - k^2\epsilon^2 - \epsilon^2 k^2 + \eta^{22} k_\alpha \epsilon^\alpha = A^{22} - k(\epsilon^0 - \epsilon^3) \tag{46}$$

In this way we have added four constraints (the values of ϵ^μ) and we can use eqs. (41)-(46) to reduce the independent components of A to just two. A^{12} remains unchanged by the gauge transformation, and $A^{21} = A^{12}$ by symmetry, and we can select just one more independent value. Setting

$$\epsilon^0 = (2A^{00} + A^{11} + A^{22})/4k \tag{47}$$

$$\epsilon^1 = A^{01}/k \tag{48}$$

$$\epsilon^2 = A^{02}/k \tag{49}$$

$$\epsilon^3 = (2A^{00} - A^{11} - A^{22})/4k \tag{50}$$

we find the following transformed A matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^{11} & A^{12} & 0 \\ 0 & A^{12} & -A^{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A^{11} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + A^{12} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{51}$$

After defining two *linear polarization matrices*

$$\epsilon_+^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad \epsilon_\times^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{52}$$

we can write the generic polarization state

$$A'^{\mu\nu} = A_+\epsilon_+^{\mu\nu} + A_\times\epsilon_\times^{\mu\nu} \tag{53}$$

Both matrices are traceless and the components in direction 3 vanishe (transverse propagation), therefore this choice of the gauge is called the **transverse-traceless gauge** (TT gauge). In this gauge $\bar{h} = 0$, therefore $h = 0$ as well and **in this gauge there is no difference between perturbation variables and trace-reversed perturbation variables**.

In linearized gravity and in the TT gauge, the formula for the connection coefficients becomes

$$\Gamma^\alpha_{\mu\nu} \approx \frac{1}{2}\eta^{\alpha\gamma}\left(\partial_\mu h_{\gamma\nu} + \partial_\nu h_{\gamma\mu} - \partial_\gamma h_{\mu\nu}\right) \tag{54}$$

$$= \frac{1}{2}\eta^{\alpha\gamma}\left(k_\mu h_{\gamma\nu} + k_\nu h_{\gamma\mu} - k_\gamma h_{\mu\nu}\right); \tag{55}$$

considering this expression, it is easy to see that

$$\Gamma^\mu_{00} = 0 \quad \Gamma^\mu_{0\nu} = \frac{1}{2}\partial_0 h^\mu_\nu.$$

Using these results, and considering a particle initially at rest, so that its initial 4-velocity is $\dot{x}^\mu = (c, 0, 0, 0)$, the geodesic equation

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0, \tag{56}$$

becomes

$$\frac{d^2x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\Gamma^\alpha_{00}c^2 = 0. \tag{57}$$

This means that the 4-velocity remains constant and equal to its initial value (particle at rest). In other words, in the TT gauge, a spherical cloud of particles at rest has geodesics with constant spatial coordinate: therefore, the small spacelike vectors $\xi^\mu = (0, \xi^1, \xi^2, \xi^3)$ that mark the separations between nearby particles in the cloud remain constant.

However, the spatial separation $\Delta\mathbf{x}^2$ is *not* constant:

$$\begin{aligned}\Delta\mathbf{x}^2 = -g_{ij}\xi^i\xi^j = -(\eta_{ij} + h_{ij})\xi^i\xi^j = (\delta_{ij} - h_{ij})\xi^i\xi^j = \xi_i\xi^i - h_{ij}\xi^i\xi^j \\ \approx \left(\xi_i - \frac{1}{2}h_{ik}\xi^k\right)\left(\xi^i - \frac{1}{2}h^i_k\xi^k\right)\end{aligned} \tag{58}$$

The new variables $\xi^i - \frac{1}{2}h^i_k\xi^k$ mark the correct spatial separation. Note that in the TT gauge, there is no shift in the 3 direction (the propagation direction), again showing that the wave is transverse.

When we take one of these particles, originally in $(\xi^1, \xi^2, 0)$ as position marker, we see that with a passing wave with amplitude $A_+\epsilon_+^{\mu\nu}$ its position is

$$x^1 = \xi^1 - \frac{A_+}{2}\cos\omega t\, \xi^1 \tag{59}$$

$$x^2 = \xi^2 + \frac{A_+}{2}\cos\omega t\, \xi^2 \tag{60}$$

$$x^3 = 0 \tag{61}$$

This means that in the $\xi^3 = 0$ plane, a ring of N equally spaced reference masses at radius R and angular position $\theta_n = 2\pi n/N$ has separation

$$r_n^2 = R^2\left(1 - \frac{A_+}{2}\cos\omega t\right)^2 \cos^2\theta_n + R^2\left(1 + \frac{A_+}{2}\cos\omega t\right)^2 \sin^2\theta_n \tag{62}$$

$$\approx R^2\left[(1 - A_+\cos\omega t)\cos^2\theta_n + (1 + A_+\cos\omega t)^2\sin^2\theta_n\right] \tag{63}$$

$$= R^2\left[1 - A_+(\cos^2\theta_n - \sin^2\theta_n)\cos\omega t\right] \tag{64}$$

$$= R^2\left(1 - A_+\cos 2\theta_n\ \cos\omega t\right), \tag{65}$$

and finally

$$r_n \approx R\left(1 - \frac{A_+}{2}\cos 2\theta_n \cos\omega t\right). \tag{66}$$

We see that the perturbation variable represents the relative deformation of the distance between test masses (see Fig. 1); in the theory of elasticity this is called a **strain**.

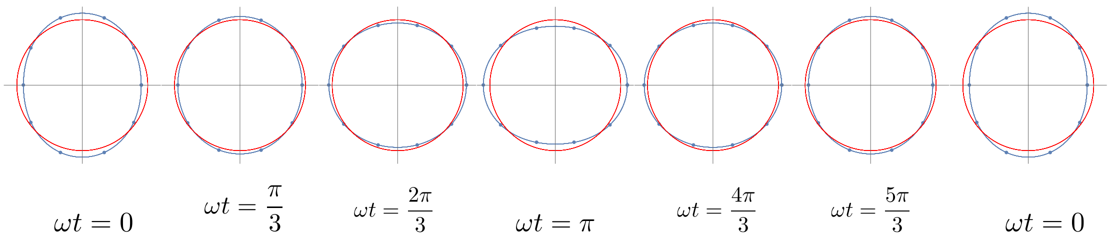

Figure 1: Spatial deformations (blue) of a circular ring of particles (shown in red due to the + polarization.