Generation of gravitational waves

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1 The quadrupole formula

Here we reconsider the wave equation with a source term (see the handout "The transverse-traceless gauge"):

$$\Box^2 \bar{h}^{\mu\nu} = \frac{16\pi G}{c^4} T^{\mu\nu}. \tag{1}$$

$$\partial_\nu \bar{h}^{\mu\nu} = 0 \tag{2}$$

Formally, eq. (1) is just like the equations for the individual electromagnetic vector potential components $A^\mu = (\phi/c, \mathbf{A})$ in the Lorentz gauge, in particular for the 0 component (the electric potential) $\phi = cA^0$ the equation is:

$$\Box^2 \phi = \frac{\rho}{\varepsilon_0} \tag{3}$$

$$\partial_\mu A^\mu = 0 \tag{4}$$

The solution of eq. (3) in vacuum, based on retarded potentials, is well-known

$$\phi(ct, \mathbf{x}) = \int_{\text{source}} \frac{\rho(ct - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{4\pi\varepsilon_0 |\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}', \tag{5}$$

where $\mathbf{r}_0$ is the position of a small volume of the source, $\mathbf{r}$ is the position where the field is determined, and $r = |\mathbf{r} - \mathbf{r}_0|$ is their distance. Since, formally, eq. (1) can be obtained from eq. (3) with the substitutions $\rho \to T^{\mu\nu}$ and $1/\varepsilon_0 \to 16\pi G/c^4$, we see that the solution of eq. (1) is

$$\bar{h}^{\mu\nu}(ct, \mathbf{x}) = \frac{4G}{c^4} \int_{\text{source}} \frac{T^{\mu\nu}(ct - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'. \tag{6}$$

Under the following conditions:

1. source size $\ll$ wavelength λ of the wave $\ll$ distance r to the source;
2. $|\bar{h}^{\mu\nu}| \ll 1$;

3. source is slow (all its parts move with speed $\ll c$);

the solution (6) approximates to

$$\bar{h}^{\mu\nu}(ct, \mathbf{x}) \approx \frac{4G}{c^4 r} \int_{\text{source}} T^{\mu\nu}(ct - r, \mathbf{x}') d^3\mathbf{x}'. \tag{7}$$

In view of the results obtained in the TT gauge, we are interested in the space part of the metric perturbation tensor, h^{ij}, and we proceed to evaluate it.

Notice that applying the Lorentz condition to both sides of eq. (1), we find

$$\Box^2(\partial_\nu \bar{h}^{\mu\nu}) = \frac{16\pi G}{c^4} \partial_\nu T^{\mu\nu} = 0, \tag{8}$$

i.e.,

$$\partial_\nu T^{\mu\nu} = 0 \tag{9}$$

(the Lorentz condition in linearized gravity is consistent with the local conservation of energy), where the time part is

$$\partial_\nu T^{0\nu} = \partial_0 T^{00} + \partial_k T^{0k} = 0, \tag{10}$$

and the space part is

$$\partial_\nu T^{i\nu} = \partial_0 T^{i0} + \partial_k T^{ik} = 0. \tag{11}$$

We use these equations to prove an identity that helps evaluating the integral (7).

We start with

$$\int_{\text{source}} \partial_k (T^{ik} x^j)\, d^3\mathbf{x} = \int_{\text{source}} \left[\partial_k T^{ik} x^j + T^{ik} \delta^j_k\right] d^3\mathbf{x}$$
$$= \int_{\text{source}} \left[-\partial_0 T^{i0} x^j + T^{ij}\right] d^3\mathbf{x}, \tag{12}$$

where the integral on the l.h.s. is a divergence, and thanks to Gauss' theorem it is equivalent to a surface integral on the boundary of the mass-energy distribution of the source; however, at the boundary, we can assume a vanishing T^{ik}, and the whole integral evaluates to zero. Therefore, using this result and the small-speed approximation,

$$\int_{\text{source}} T^{ij}\, d^3\mathbf{x} = \int_{\text{source}} \partial_0 T^{i0} x^j\, d^3\mathbf{x} \approx \frac{1}{c} \frac{d}{dt} \int_{\text{source}} T^{i0} x^j\, d^3\mathbf{x}, \tag{13}$$

and exchanging indexes and summing, we find

$$\int_{\text{source}} T^{ij}\, d^3\mathbf{x} \approx \frac{1}{2c} \frac{d}{dt} \int_{\text{source}} \left(T^{i0} x^j + T^{j0} x^i\right) d^3\mathbf{x}. \tag{14}$$

The r.h.s. of this equation can be further transformed as follows. First, we remark that

$$\int_{\text{source}} \partial_k \left(T^{0k} x^i x^j\right) d^3\mathbf{x} = \int_{\text{source}} \left(\partial_k T^{0k} x^i x^j + T^{0k} x^i \delta^j_k + T^{0k} x^j \delta^i_k\right) d^3\mathbf{x}$$
$$= \int_{\text{source}} \partial_k T^{0k} x^i x^j \; d^3\mathbf{x} + \int_{\text{source}} \left(T^{0j} x^i + T^{0i} x^j\right) d^3\mathbf{x}, \quad (15)$$

where the l.h.s. volume integral has an integrand which is a divergence and can be transformed into a surface integral that vanishes as above. Therefore,

$$\int_{\text{source}} \left(T^{0j} x^i + T^{0i} x^j\right) d^3\mathbf{x} = -\int_{\text{source}} \partial_k T^{0k} x^i x^j dV = \int_{\text{source}} \partial_0 T^{00} x^i x^j \; d^3\mathbf{x}, \quad (16)$$

where we used eq. (10) in the last passage, and finally, from eqs. (14) and (16) we find

$$\int_{\text{source}} T^{ij} \; d^3\mathbf{x} \approx \frac{1}{2c^2} \frac{d^2}{dt^2} \int_{\text{source}} T^{00} x^i x^j \; d^3\mathbf{x}. \quad (17)$$

We can use this result and the slow-motion assumption $T^{00} \approx \rho c^2$, where ρ is the mass density, to obtain

$$\bar{h}^{ij}(ct, \mathbf{r}) \approx \frac{4G}{c^4 r} \int_{\text{source}} T^{ij}(ct - r, \mathbf{r}) \; d^3\mathbf{x}$$
$$= \frac{4G}{c^4 r} \frac{1}{2c^2} \frac{d^2}{dt^2} \left[\int_{\text{source}} T^{00} x^i x^j \; d^3\mathbf{x}\right]_{\text{retarded}} = \frac{2G}{c^4 r} \frac{d^2}{dt^2} \left[\int_{\text{source}} \rho x^i x^j \; d^3\mathbf{x}\right]_{\text{retarded}}, \quad (18)$$

where the integrals are evaluated at the retarded time. The integral

$$Q^{ij} = \int_{\text{source}} \rho x^i x^j \; d^3\mathbf{x} \quad (19)$$

is the *quadrupole tensor* of the mass distribution, so that the solution can also be written in the form

$$\bar{h}^{ij}(ct, \mathbf{x}) \approx \frac{2G}{c^4 r} \ddot{Q}^{ij} \left(t - r/c\right). \quad (20)$$

This equation is the famous *quadrupole formula.* **The first-order term in GW generation is due to the mass-energy quadrupole. There is no monopole or dipole term.**

2 No GWs from a spherically symmetric source

In the TT gauge, the strain is traceless: this shows that only the off-diagonal terms of the quadrupole tensor contribute to the emission of gravitational waves. However, for a radially symmetric mass distribution we can write eq. (19) in polar form

$$Q^{ij} = \int_0^\infty \rho(r) r^2 dr \int_\Omega d\Omega \; n^i n^j \quad (21)$$

where

$$n^1 = \sin\theta\cos\phi;\ n^2 = \sin\theta\sin\phi;\ n^3 = \cos\theta. \tag{22}$$

Therefore, all possible off-diagonal products have vanishing integral over ϕ, and we find that the only non-vanishing elements of the Q^{ij} tensor lie on the diagonal. **This means that spherically symmetric motions do not produce any gravitational radiation.**

Indeed, in a previous handout, we found that there is a very useful specialization of the Lorentz gauge, the TT gauge, a coordinate system that is comoving with the wave itself. In the TT gauge, free particles remain at constant coordinate locations, although their proper separations change. We find the amplitude of the GW in the TT gauge by projecting the quadrupole tensor in the plane perpendicular to the direction of the wave, and by removing the trace of the projected tensor (recall that the polarization tensors are both traceless).

Finally, we note that the nonspherical part of the quadrupole tensor is given by the *reduced quadrupole tensor*

$$\rlap{-}I^{ij} = \int_{\text{source}} \rho(\mathbf{x}) \left(x^i x^j + \frac{1}{3}\eta^{ij}|\mathbf{x}|^2 \right) d^3\mathbf{x}, \tag{23}$$

so that the quadrupole formula, Eq. (20), becomes

$$\bar{h}_{ij} \approx \frac{2G}{c^4 r} \frac{d^2 \rlap{-}I_{ij}}{dt^2} \tag{24}$$

Both the quadrupole tensor and the reduced quadrupole tensor are close relatives of the classical inertia tensor which is defined by

$$I^{ij} = \int_{\text{source}} \rho(\mathbf{x}) \left(-\eta^{ij}|\mathbf{x}|^2 - x^i x^j \right) d^3\mathbf{x}, \tag{25}$$

3 GWs radiated by a rotating dumbbell

As an application of eq. (18), consider a dumbbell, with two masses M at the end of a massless rod of length $2R$, as illustrated in figure 1. The dumbbell rotates in the (x^1, x^2) plane with constant angular speed ω about its midpoint.

Setting the origin of the coordinate system at the midpoint of the rod, the positions of the masses are

$$x^i = \pm(R\cos\omega t, R\sin\omega t, 0) \tag{26}$$

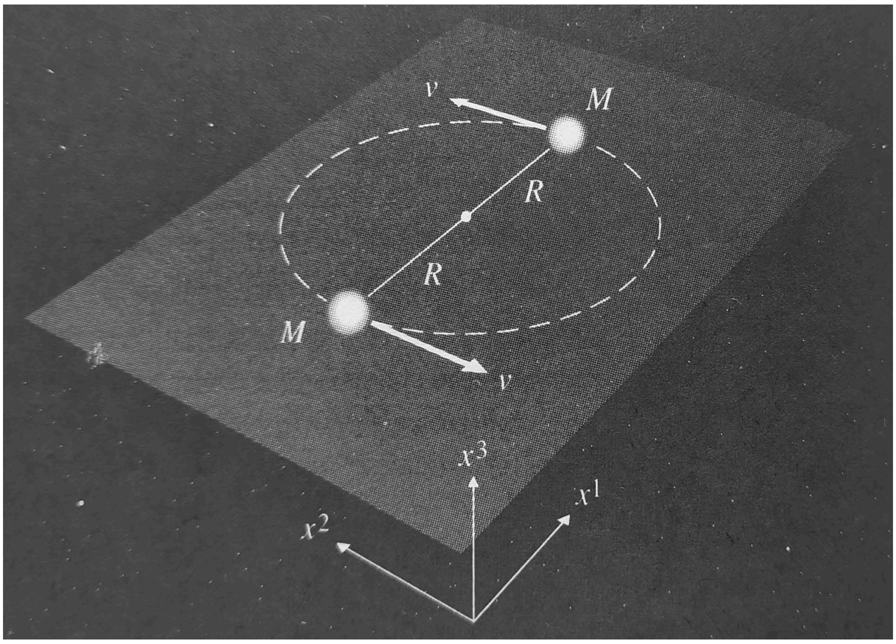

Figure 1: Example of a “dumbbell”. In this case the stars in a binary system orbit one around the other in a circle of radius R (figure from S. M. Carroll, *Spacetime Geometry, an Introduction to General Relativity* Pearson, 2013).

and we find

$$[\bar{h}^{ij}(ct,\mathbf{r})] = -\frac{4GMR^2}{c^4 r}\,\frac{d^2}{dt^2}\begin{pmatrix} \cos^2\omega t & \cos\omega t\sin\omega t & 0 \\ \cos\omega t\sin\omega t & \sin^2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\text{retarded}} \tag{27}$$

$$= \frac{8GMR^2\omega^2}{c^4 r}\begin{pmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}_{\text{retarded}} \tag{28}$$

$$= \frac{8GMR^2\omega^2}{c^4 r}\begin{pmatrix} \cos 2\omega(t-r/c) & \sin 2\omega(t-r/c) & 0 \\ \sin 2\omega(t-r/c) & -\cos 2\omega(t-r/c) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{29}$$

This solution represents a gravitational wave with frequency 2ω. Both polarization components are equally represented and they are 90° out of phase, so that this is a circularly polarized gravitational wave.

It is easy to see that in the case of unequal masses M_1, M_2, and radii R_1, R_2, the previous result becomes

$$[\bar{h}^{ij}(ct,\mathbf{r})] = \frac{4G(M_1R_1^2 + M_2R_2^2)\omega^2}{c^4 r} \begin{pmatrix} \cos 2\omega(t - r/c) & \sin 2\omega(t - r/c) & 0 \\ \sin 2\omega(t - r/c) & -\cos 2\omega(t - r/c) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{30}$$

We can use these results to find the order of magnitude of the amplitude of waves emitted by a laboratory-size apparatus. We assume here $M = 1$ kg, $R = 1$ m, $\omega = 1\ \mathrm{s}^{-1}$. Moreover, to satisfy the far-field approximation, $r \gg c/\omega$, then the amplitude of the gravitational waves is

$$h \approx \frac{8GMR^2\omega^2}{c^4 r} \ll \frac{8GMR^2\omega^3}{c^5} \sim 10^{-52}, \tag{31}$$

which is totally undetectable with current technologies.

3.1 Binary systems

In the case of a binary star system with a large separation between stars (which the dumbbell with vanishing-mass rod approximates) and circular orbits, we can use the Keplerian formulas to obtain the frequency ω. With masses m_1 and m_2, with orbital radii r_1 and r_2, in the CM system the total momentum vanishes and we find

$$m_1 r_1 \omega = m_2 r_2 \omega \tag{32}$$

i.e.,

$$m_1 r_1 = m_2 r_2 \tag{33}$$

Moreover, the masses experience a centrifugal acceleration that must be balanced by the gravitational force, so that

$$m_1 r_1 \omega^2 = m_2 r_2 \omega^2 = \frac{G m_1 m_2}{(r_1 + r_2)^2} \tag{34}$$

The latter formula can be rearranged to obtain

$$r_1 \omega^2 = \frac{G m_2}{(r_1 + r_2)^2}; \quad r_2 \omega^2 = \frac{G m_1}{(r_1 + r_2)^2} \tag{35}$$

and therefore

$$\omega^2 = \frac{G(m_1 + m_2)}{(r_1 + r_2)^3} \tag{36}$$

(a form of Kepler's third law). Therefore, the closer the stars, the higher the frequency, and therefore the larger the metric perturbation. **As a result, we expect measurable GWs from very compact binary systems, typically formed by close pairs of black holes or neutron stars.** We shall develop these ideas further when discussing gravitational radiation from compact binary systems.