The stress-energy tensor of gravitational waves and the energy flux

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We already know that in linearized gravity, and taking the Lorentz gauge, the Einstein equation is

$$2G^{(1)}_{\mu\nu} = \Box^2 \bar{h}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu} \tag{1}$$

where $G^{(1)}_{\mu\nu}$ denotes the first-order term of the Einstein tensor and $T_{\mu\nu}$ is the non-gravitational stress-energy tensor. Considering the second-order term as well, we find

$$2G^{(1)}_{\mu\nu} + 2G^{(2)}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu} \tag{2}$$

and we can consider the second-order term as something new that takes into account the self-interaction of the gravitational field with itself, and in particular we can write

$$\Box^2 \bar{h}_{\mu\nu} = \frac{16\pi G}{c^4} \left(T_{\mu\nu} + T^{\mathrm{GW}}_{\mu\nu} \right) \tag{3}$$

where

$$T^{\mathrm{GW}}_{\mu\nu} = -\frac{c^4}{8\pi G} G^{(2)}_{\mu\nu} \tag{4}$$

is the gravitational contribution to the stress-energy tensor.

Recalling that in the Lorentz gauge $\partial_\nu \bar{h}^{\mu\nu} = 0$, we obtain

$$\partial^\nu \left(T_{\mu\nu} + T^{\mathrm{GW}}_{\mu\nu} \right) = 0 \tag{5}$$

so that locally, the total energy is conserved.

Actually, energy conservation holds only in flat space, thus the previous energy-conservation formula only holds when averaged over several GW wavelenghts, and the proper definition of the gravitational stress-energy tensor is

$$T^{\mathrm{GW}}_{\mu\nu} = \frac{c^4}{8\pi G} \left\langle G^{(2)}_{\mu\nu} \right\rangle \tag{6}$$

where the average is taken over several GW wavelengths, in the weak field limit.

In a dedicated handout (*The TT metric GW worksheet*) we find that the energy density (tt component of the stress-energy tensor) can be written in the form

$$T_{tt}^{\mathrm{GW}} = \frac{c^4}{8\pi G}\left\langle G_{tt}^{(2)} \right\rangle = \frac{c^2}{16\pi G}\left\langle \dot{h}_+ \dot{h}_+ \right\rangle, \tag{7}$$

and it is quite obvious that the same result must hold for the rotated polarization, so that, overall, the energy density is

$$T_{tt}^{\mathrm{GW}} = \frac{c^2}{16\pi G}\left\langle \dot{h}_+ \dot{h}_+ + \dot{h}_\times \dot{h}_\times \right\rangle = \frac{c^2}{32\pi G}\left\langle \dot{h}_{jk} \dot{h}^{jk} \right\rangle, \tag{8}$$

where the rightmost form of the equation is written in such a way that it holds for any spatial direction[1]. A simple argument shows that the energy flux of the gravitational wave is equal to

$$\text{energy flux} = c\, T_{tt}^{\mathrm{GW}} = \frac{c^3}{32\pi G}\left\langle \dot{h}_{jk} \dot{h}^{jk} \right\rangle. \tag{9}$$

Now, recall the equation that we obtained in the *Generation of gravitational waves* handout

$$\bar{h}_{ij} \approx \frac{2G}{c^4 r}\frac{d^2 I\!\!\!-_{ij}^{TT}}{dt^2}, \tag{10}$$

then

$$\text{energy flux} = \frac{c^3}{32\pi G}\frac{4G^2}{c^8 r^2}\left\langle \dddot{I\!\!\!-}_{ij}^{TT} \dddot{I\!\!\!-}_{TT}^{ij} \right\rangle = \frac{G}{8\pi c^5}\frac{\left\langle \dddot{I\!\!\!-}_{ij}^{TT} \dddot{I\!\!\!-}_{TT}^{ij} \right\rangle}{r^2} \tag{11}$$

where the TT label reminds us that this result has been obtained in the TT gauge.

[1] The factor $\dot{h}_{jk}\dot{h}^{jk}$ is the trace of a diagonal matrix, and is equal to $2(\dot{h}_+\dot{h}_+ + \dot{h}_\times\dot{h}_\times)$.