The stress-energy tensor of gravitational waves and the energy flux

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We already know that in linearized gravity, and taking the Lorentz gauge, the Einstein equation is

$$2G^{(1)}_{\mu\nu} = \Box^2 \bar{h}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu}, \tag{1}$$

(see the handout "Linearized gravity") where $G^{(1)}_{\mu\nu}$ denotes the first-order term of the Einstein tensor and $T_{\mu\nu}$ is the non–gravitational stress–energy tensor. Considering the second–order term that takes into account the stress–energy of a gravitational wave we write

$$2G^{(1)}_{\mu\nu} + 2G^{(2)}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu}, \tag{2}$$

i.e.,

$$2G^{(1)}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu} - 2G^{(2)}_{\mu\nu}, \tag{3}$$

and we obtain

$$\Box^2 \bar{h}_{\mu\nu} = 2G^{(1)}_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu} - 2G^{(2)}_{\mu\nu} = \frac{16\pi G}{c^4} \left(T_{\mu\nu} + T^{\mathrm{GW}}_{\mu\nu} \right), \tag{4}$$

where

$$T^{\mathrm{GW}}_{\mu\nu} = -\frac{c^4}{8\pi G} G^{(2)}_{\mu\nu} \tag{5}$$

is the gravitational contribution to the stress-energy tensor.

Recalling that in the Lorentz gauge $\partial_\nu \bar{h}^{\mu\nu} = 0$, we obtain

$$\partial^\nu \left(T_{\mu\nu} + T^{\mathrm{GW}}_{\mu\nu} \right) = 0, \tag{6}$$

so that locally, the total energy is conserved.

Actually, energy conservation holds only in flat space, thus the previous energy-conservation formula only holds when averaged over several GW wavelenghts, and the proper definition of the gravitational stress-energy tensor is

$$T^{\mathrm{GW}}_{\mu\nu} = \frac{c^4}{8\pi G} \left\langle G^{(2)}_{\mu\nu} \right\rangle, \tag{7}$$

where the average is taken over several GW wavelengths, in the weak field limit.

Evaluating the second order term of the Einstein tensor for gravitational waves means computing its component tensors, and in particular the Ricci tensor. The complete calculation is carried out in a dedicated handout, "The TT metric GW worksheet", where we find that the energy density (tt component of the stress-energy tensor) can be written in the form

$$T_{tt}^{\mathrm{GW}} = \frac{c^4}{8\pi G}\left\langle G_{tt}^{(2)}\right\rangle = \frac{c^2}{16\pi G}\left\langle \dot{h}_+\dot{h}_+\right\rangle, \tag{8}$$

and it is quite obvious that the same result must hold for the rotated polarization, so that, overall, the energy density is

$$T_{tt}^{\mathrm{GW}} = \frac{c^2}{16\pi G}\left\langle \dot{h}_+\dot{h}_+ + \dot{h}_\times\dot{h}_\times\right\rangle = \frac{c^2}{32\pi G}\left\langle \dot{h}_{jk}\dot{h}^{jk}\right\rangle, \tag{9}$$

where the rightmost form of the equation is written in such a way that it holds for any spatial direction[1]. A simple argument shows that the energy flux of the gravitational wave is equal to

$$\text{energy flux} = c\, T_{tt}^{\mathrm{GW}} = \frac{c^3}{32\pi G}\left\langle \dot{h}_{jk}\dot{h}^{jk}\right\rangle. \tag{10}$$

Now, recall the equation that we obtained in the handout "Generation of gravitational waves"

$$\bar{h}_{ij} \approx \frac{2G}{c^4 r}\,\frac{d^2 \rlap{-}I_{ij}^{TT}}{dt^2}, \tag{11}$$

then

$$\text{energy flux} = \frac{c^3}{32\pi G}\,\frac{4G^2}{c^8 r^2}\left\langle \dddot{\rlap{-}I}{}_{ij}^{TT}\,\dddot{\rlap{-}I}{}_{TT}^{ij}\right\rangle = \frac{G}{8\pi c^5}\,\frac{\left\langle \dddot{\rlap{-}I}{}_{ij}^{TT}\,\dddot{\rlap{-}I}{}_{TT}^{ij}\right\rangle}{r^2} \tag{12}$$

where the TT label reminds us that this result has been obtained in the TT gauge.

[1] The factor $\dot{h}_{jk}\dot{h}^{jk}$ is the trace of a diagonal matrix, and is equal to $2(\dot{h}_+\dot{h}_+ + \dot{h}_\times\dot{h}_\times)$.