The TT metric GW worksheet

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November 11, 2024

To carry out the calculations we need to recall a few important expressions:

- Christoffel symbols

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta} \right) \tag{1}$$

- Riemann tensor

$$R^{\mu}_{\alpha\beta\gamma} = \partial_\beta \Gamma^{\mu}_{\alpha\gamma} - \partial_\gamma \Gamma^{\mu}_{\alpha\beta} + \Gamma^{\delta}_{\alpha\gamma} \Gamma^{\mu}_{\delta\beta} - \Gamma^{\delta}_{\alpha\beta} \Gamma^{\mu}_{\delta\gamma} \tag{2}$$

- Ricci tensor

$$R_{\alpha\beta} = R^{\mu}_{\alpha\beta\mu} = \partial_\beta \Gamma^{\mu}_{\alpha\mu} - \partial_\mu \Gamma^{\mu}_{\alpha\beta} + \Gamma^{\delta}_{\alpha\mu} \Gamma^{\mu}_{\delta\beta} - \Gamma^{\delta}_{\alpha\beta} \Gamma^{\mu}_{\delta\mu} \tag{3}$$

- metric tensor for the + polarization

$$[g_{\mu\nu}] = [\eta^{\mu\nu}] + [h^{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 + h(t,z) & 0 & 0 \\ 0 & 0 & -1 - h(t,z) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4}$$

- inverse metric tensor for the + polarization

$$[g^{\mu\nu}] = [\eta^{\mu\nu}] - [h^{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 - h(t,z) & 0 & 0 \\ 0 & 0 & -1 + h(t,z) & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{5}$$

Next, assuming a sinusoidal time dependence

$$h(t,z) = A \cos(\omega t - kz) = h^{TT}_{xx} = -h^{TT}_{yy} \tag{6}$$

and using the *Diagonal metric worksheet* of T. A. Moore[1] we carry out the calculations that show that for the + polarization

$$R_{tt} = -\frac{1}{c^2}\left(h\ddot{h} + \frac{\dot{h}^2}{2}\right); \quad R_{zz} = R_{tt}; \quad R_{xx} = R_{yy} = 0 \tag{7}$$

to second order in h.

After averaging over time, we find

$$\begin{aligned}
-\left\langle h\ddot{h} + \frac{\dot{h}^2}{2}\right\rangle &= \omega^2 A^2 \left\langle \cos^2(\omega t - kz) - \frac{1}{2}\sin^2(\omega t - kz)\right\rangle \\
&= \omega^2 A^2 \left\langle \cos^2(\omega t - kz) - \sin^2(\omega t - kz)\right\rangle + \omega^2 A^2 \left\langle \frac{1}{2}\sin^2(\omega t - kz)\right\rangle \\
&= \omega^2 A^2 \left\langle \cos(2\omega t - 2kz)\right\rangle + \frac{1}{2}\left\langle \dot{h}^2\right\rangle = \frac{1}{2}\left\langle \dot{h}^2\right\rangle
\end{aligned} \tag{8}$$

Using equations (7) and recalling the form of the metric, we can easily prove that for the gravitational wave $R = 0$. Therefore

$$\left\langle G^{(2)}_{tt}\right\rangle = \left\langle R^{(2)}_{tt}\right\rangle = \frac{1}{2}\left\langle \dot{h}^2\right\rangle \tag{9}$$

[1] We must be careful that in Moore's text he uses the (-,+,+,+) signature while we use the (+,-,-,-) signature, and units such that $c = 1$. He also uses a different convention for the Ricci tensor, so that the sign is reversed with respect to our definition.