Compact Binary Coalescences

Edoardo Milotti

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1 Introduction

To begin with, we partly repeat the calculation we already did to find the strain produced by a rotating dumbbell, and apply the formula for the total emitted power. We consider a pair of stars with equal masses m in circular orbit about their common CM with orbital radius r, as illustrated in figure 1 (note however, that in the main text we use m instead of M and r instead of R). In this simple scheme, the stars rotate in the (x^1, x^2) plane with constant angular speed ω .



Figure 1: A binary star system. In this case the stars in the binary system orbit one around the other in a circle of radius R (figure from S. M. Carroll, *Spacetime Geometry*, an Introduction to General Relativity Pearson, 2013).

Setting the origin of the coordinate system at the CM, the positions of the masses are

$$x^{i} = \pm (r \cos \omega t, r \sin \omega t, 0) \tag{1}$$

and we find the quadrupole tensor

$$[Q^{ij}] = 2mr^2 \begin{pmatrix} \cos^2 \omega t & \cos \omega t \sin \omega t & 0\\ \cos \omega t \sin \omega t & \sin^2 \omega t & 0\\ 0 & 0 & 0 \end{pmatrix}$$
(2)

and the reduced quadrupole tensor

$$[\mathcal{F}^{ij}] = 2mr^2 \begin{pmatrix} \cos^2 \omega t - 1/3 & \cos \omega t \sin \omega t & 0\\ \cos \omega t \sin \omega t & \sin^2 \omega t - 1/3 & 0\\ 0 & 0 & -1/3 \end{pmatrix}$$
(3)

Then

$$\begin{bmatrix} \ddot{\mathcal{F}}^{ij} \end{bmatrix} = 2mr^2 \begin{pmatrix} 4\omega^3 \sin 2\omega t & -4\omega^3 \cos 2\omega t & 0\\ -4\omega^3 \cos 2\omega t & -4\omega^3 \sin 2\omega t & 0\\ 0 & 0 & 0 \end{pmatrix}$$
$$= 8\omega^3 mr^2 \begin{pmatrix} \sin 2\omega t & -\cos 2\omega t & 0\\ -\cos 2\omega t & -\sin 2\omega t & 0\\ 0 & 0 & 0 \end{pmatrix}$$
(4)

(as an exercise, fill in all the missing details in this calculation). Therefore,

$$\left\langle \vec{\mu}^{mn} \, \vec{\mu}_{mn} \right\rangle = 128 \, m^2 r^4 \omega^6, \tag{5}$$

and, finally, we find that this system emits energy as GWs at the rate

$$P_{\rm GW} = \frac{G}{5c^5} \left\langle \ddot{\mu}^{mn} \ddot{\mu}_{mn} \right\rangle = \frac{128G}{5c^5} m^2 r^4 \omega^6 \tag{6}$$

It can be shown (do it as an exercise), that for unequal masses m_1 and m_2 , this equation can be written in the form

$$P_{\rm GW} = \frac{32G}{5c^5} \frac{m_1^2 m_2^2}{(m_1 + m_2)^2} (r_1 + r_2)^4 \omega^6 \tag{7}$$

2 Keplerian formulas for the compact binary system

We are now interested in the fate of a close pair of compact objects: as they emit energy in the form of GWs, they do this at the expense of potential energy and they gradually approach, speeding up to higher and higher orbital angular frequencies. The initially smooth and stable orbit eventually becomes a frantic race until the two object coalesce into one. This process is well illustrated by a sketch drawn several years ago by Kip Thorne (figure 2).

In this handout, we develop a simplified Newtonian model of the inspiral phase, as in refs. [3, 1].



Figure 2: Drawing by Kip Thorne that illustrates the main phases of a compact binary coalescence (CBC). Initially the pair rotates at a nearly constant frequency, but at a later phase the energy loss due to GW emission is large and increasing and the orbital frequency increases faster and faster while the objects become closer and closer (inspiral phase). This continues until the merger phase, were the objects merger into one. Finally, the remnant object rings much like a bell with damped oscillations (ringdown phase).

We consider a non-relativistic two-body system with circular orbits about the centerof-mass (CM). The masses are m_1 and m_2 , with orbital radii r_1 and r_2 . In the CM system the total momentum vanishes, and therefore

$$m_1 r_1 \omega = m_2 r_2 \omega \tag{8}$$

i.e.,

$$m_1 r_1 = m_2 r_2$$
 (9)

The masses also experience a centrifugal acceleration that must be balanced by the

gravitational force, so that

$$m_1 r_1 \omega^2 = m_2 r_2 \omega^2 = \frac{G m_1 m_2}{(r_1 + r_2)^2} \tag{10}$$

The latter formula can be rearranged to obtain

$$r_1\omega^2 = \frac{Gm_2}{(r_1 + r_2)^2}; \quad r_2\omega^2 = \frac{Gm_1}{(r_1 + r_2)^2}$$
 (11)

and therefore

$$\omega^2 = \frac{G(m_1 + m_2)}{(r_1 + r_2)^3} \tag{12}$$

(a form of Kepler's third law), so that we can express the sum of radii as a function of total mass and frequency

$$r_1 + r_2 = \left(\frac{G(m_1 + m_2)}{\omega^2}\right)^{1/3} \tag{13}$$

Finally, the total energy of the system is

$$E_{\text{tot}} = \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 - \frac{Gm_1m_2}{(r_1 + r_2)}$$
(14)

Now notice that the moment of inertia of the system about the CM is, using eq. (9),

$$I = m_1 r_1^2 + m_2 r_2^2 = m_2 r_1 r_2 + m_1 r_1 r_2 = (m_1 + m_2) r_1 r_2$$
(15)

$$=\frac{(m_1+m_2)^2}{m_1+m_2}r_1r_2\tag{16}$$

$$=\frac{m_1^2 r_1 r_2 + m_2^2 r_1 r_2 + 2m_1 m_2 r_1 r_2}{m_1 + m_2} \tag{17}$$

$$=\frac{m_1m_2r_2^2 + m_1m_2r_1^2 + 2m_1m_2r_1r_2}{m_1 + m_2} \tag{18}$$

$$=\frac{m_1m_2}{m_1+m_2}(r_1+r_2)^2\tag{19}$$

Therefore

$$E_{\text{tot}} = \frac{1}{2}I\omega^2 - \frac{Gm_1m_2}{(r_1 + r_2)} = \frac{1}{2}\frac{Gm_1m_2}{(r_1 + r_2)} - \frac{Gm_1m_2}{(r_1 + r_2)} = -\frac{1}{2}\frac{Gm_1m_2}{(r_1 + r_2)}$$
(20)

or also

$$E_{\rm tot} = -\frac{1}{2} G^{2/3} \frac{m_1 m_2}{(m_1 + m_2)^{1/3}} \omega^{2/3}$$
(21)

3 GW strain

We found earlier (see the handout on *Generation of gravitational waves*) that the strain in the TT gauge of such a binary system is

$$[\bar{h}^{ij}(ct,\mathbf{r})] = \frac{4G(m_1r_1^2 + m_2r_2^2)\omega^2}{c^4D} \begin{pmatrix} \cos(2\omega t + \phi) & \sin(2\omega t + \phi) & 0\\ \sin(2\omega t + \phi) & -\cos(2\omega t + \phi) & 0\\ 0 & 0 & 0 \end{pmatrix}.$$
 (22)

where D is the distance to the system and ϕ is a phase that includes retarded time and the integration constant of the equations of motion of the pair of masses.

Using eqs. (10) and (13) we find

$$(m_1 r_1^2 + m_2 r_2^2)\omega^2 = G^{2/3} \frac{m_1 m_2}{(m_1 + m_2)^{1/3}} \omega^{2/3}$$
(23)

When dealing with compact objects like black holes or neutron stars it is useful to write the masses in units of solar mass $M_{\odot} \approx 2 \times 10^{30}$ kg. Then, it turns out that the combination

$$\frac{GM_{\odot}}{c^3} \approx 5 \times 10^{-6} \text{N m}^{-1} \text{kg}^{-1} \text{s}^3$$
(24)

pops up quite often. Then we obtain

$$[\bar{h}^{ij}(ct,\mathbf{r})] = \frac{4c}{D} \left(\frac{GM_{\odot}}{c^3}\right)^{5/3} \frac{m_1 m_2}{(m_1 + m_2)^{1/3}} \omega^{2/3} \left(\begin{array}{c} \cos(2\omega t + \phi) & \sin(2\omega t + \phi) & 0\\ \sin(2\omega t + \phi) & -\cos(2\omega t + \phi) & 0\\ 0 & 0 & 0\end{array}\right).$$
(25)

where we use units of solar mass for the masses m_1 and m_2 .

We also note that the frequency of the emitted gravitational radiation is twice the orbital frequency.

4 Shrinking of the orbit because of GW emission - 1

As the system emits gravitational waves its total energy decreases and the orbit shrinks. To calculate this shrinkage we compute the instantaneous radiated power, and to this end we replace the expression of the moment of inertia and that of its rotation frequency to eliminate both the moment of inertia and the spatial separation between the masses in the formula for the radiated power:

$$P_{\rm GW} = \frac{32G^{7/3}}{5c^5} \frac{(m_1m_2)^2}{(m_1 + m_2)^{2/3}} \,\omega^{10/3} \tag{26}$$

This radiated power corresponds to the rate of change of the total energy of the binary system

$$\left|\frac{dE_{\text{tot}}}{dt}\right| = \frac{G^{2/3}}{3} \frac{m_1 m_2}{(m_1 + m_2)^{1/3}} \,\omega^{-1/3} \,\frac{d\omega}{dt} \tag{27}$$

and equating the two expressions we find

$$\frac{96G^{5/3}}{5c^5} \frac{m_1 m_2}{(m_1 + m_2)^{1/3}} = \omega^{-11/3} \frac{d\omega}{dt}$$
(28)

or also

$$\frac{m_1 m_2}{(m_1 + m_2)^{1/3}} = \frac{c^5}{G^{5/3}} \frac{5}{96} \ \omega^{-11/3} \ \frac{d\omega}{dt}$$
(29)

It is customary to define the "chirp mass" as follows:

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} = \frac{c^3}{G} \left(\frac{5}{96} \ \omega^{-11/3} \ \frac{d\omega}{dt}\right)^{3/5} \tag{30}$$

Then, using the chirp mass and the emitted frequency $f = \omega/\pi$ we can write

$$\mathcal{M} = \frac{c^3}{G} \left(\frac{5}{96} \pi^{-8/3} f^{-11/3} \frac{df}{dt} \right)^{3/5}$$
(31)

This equation relates time and frequency, and is the only equation in the "GW150914 discovery paper".

5 Further estimates

Obviously, we can also integrate the differential equation and find

$$\frac{256}{5}\pi^{8/3}\mathcal{M}^{5/3}\frac{G^{5/3}}{c^5}(t-t_0) = f_0^{-8/3} - f^{-8/3}$$
(32)

where t_0 is the coalescence time and f_0 is the frequency at coalescence time (see below). Therefore,

$$f(t) = \left[f_0^{-8/3} - \frac{256}{5} \pi^{8/3} \mathcal{M}^{5/3} \frac{G^{5/3}}{c^5} (t - t_0) \right]^{-3/8}$$
(33)

for $t < t_0$ (times before coalescence). Eq. (33) can be expressed with the chirp mass in units of solar mass as before

$$f(t) = \left[f_0^{-8/3} - \frac{256}{5} \pi^{8/3} \left(\frac{\mathcal{M}}{M_{\odot}} \right)^{5/3} \left(\frac{GM_{\odot}}{c^3} \right)^{5/3} (t - t_0) \right]^{-3/8}$$
(34)



Figure 3: Time-frequency representation of the GW signal, as described by eq. (33), for $m_1 = m_2 = 30 M_{\odot}$. The coalescence time is $t_0 = 0$.

Figure 3 shows an example of frequency evolution for a system with $m_1 = m_2 = 30 M_{\odot}$. The corresponding signal in the time domain is shown in figure 4 (+ polarization only) and in figure 5 (both + and × polarizations).

Neglecting the constant terms in eq. (33) we find

$$\frac{256}{5}\pi^{8/3}\mathcal{M}^{5/3}\frac{G^{5/3}}{c^5}|t| \approx f^{-8/3} \tag{35}$$

or also

$$\mathcal{M} \approx \frac{c^3}{G} \, \frac{1}{(256|t|/5)^{3/5} \, (\pi f)^{8/5}} \tag{36}$$

The maximum frequency f_0 in the merger can be estimated assuming that the distance between the centers of the two masses is equal to the sum of the Schwartzschild radii (in the case of a BBH), i.e.,

$$r_1 + r_2 = \frac{2G}{c^2}(m_1 + m_2) \tag{37}$$

where $R_S = (2G/c^2)m$ is the Schwartzschild radius for a mass m. At this separation the frequency and the emitted power are at a maximum

$$\omega_0^2 = \pi^2 f_0^2 = \frac{G(m_1 + m_2)}{(r_1 + r_2)^3} = \frac{c^6}{8G^2} \frac{1}{(m_1 + m_2)^2}$$
(38)

¹Recall that the emitted frequency is twice the orbital frequency.



Figure 4: GW signal in the Newtonian approximation, for $m_1 = m_2 = 30 M_{\odot}$ and phase at coalescence $\phi_0 = 1$ radians, + polarization only. The coalescence time is $t_0 = 0$. The amplitude is scaled to its maximum.

or also

$$f_0 = \frac{1}{2\pi\sqrt{2}} \left(\frac{GM_{\odot}}{c^3}\right)^{-1} \frac{1}{M/M_{\odot}}$$
(39)

where $M = m_1 + m_2$ is the total mass of the system which is directly related to the maximum frequency of the chirp.

Since

$$\mathcal{M}^5 = \frac{(m_1 m_2)^3}{M} \tag{40}$$

we can use the observed chirp mass and total mass to obtain the system of equations

$$m_1 m_2 = M^{1/3} \mathcal{M}^{5/3} \tag{41a}$$

$$m_1 + m_2 = M \tag{41b}$$

from which we find the individual masses

$$m_{1,2} = \frac{M}{2} \pm \frac{1}{2}\sqrt{M^2 - 4\mathcal{M}^{5/3}M^{1/3}},\tag{42}$$

with the "mass gap"

$$\Delta M = \sqrt{M^2 - 4\mathcal{M}^{5/3}M^{1/3}}.$$
(43)



Figure 5: GW signal in the Newtonian approximation, for $m_1 = m_2 = 30 M_{\odot}$ and phase at coalescence $\phi_0 = 1$ radians, both + (blue curve) and × (orange curve) polarizations. The coalescence time is $t_0 = 0$. The amplitude is scaled to its maximum.

The radiated power as a function of time can also be estimated

$$\omega \approx 2 \left(\frac{256}{5} \mathcal{M}^{5/3} \frac{G^{5/3}}{c^5} t \right)^{-3/8} \sim t^{-3/8}$$
(44)

$$P_{\rm GW} = \frac{32G^{7/3}}{5c^5} \frac{(m_1m_2)^2}{(m_1 + m_2)^{2/3}} \,\omega^{10/3} \sim t^{-5/4} \tag{45}$$

6 Shrinking of the orbit because of GW emission - 2

From Kepler's third law

$$\omega^2 = \frac{GM}{R^3} \tag{46}$$

where $M = m_1 + m_2$ and $R = r_1 + r_2$, we find

$$2\omega \frac{d\omega}{dt} = -3 \frac{GM}{R^4} \frac{dR}{dt}$$
(47)

or also

$$\frac{dR}{dt} = -\frac{2}{3}\frac{R^4}{GM}\omega\frac{d\omega}{dt} = -\frac{2}{3}\frac{R^{5/2}}{\sqrt{GM}}\frac{d\omega}{dt}$$
(48)

Recall that earlier we found

$$\mathcal{M}^{5/3} = \frac{m_1 m_2}{(m_1 + m_2)^{1/3}} = \frac{c^5}{G^{5/3}} \frac{5}{96} \,\omega^{-11/3} \,\frac{d\omega}{dt} \tag{49}$$

and therefore

$$\frac{d\omega}{dt} = \frac{96\mathcal{M}^{5/3}G^{5/3}}{5c^5}\omega^{11/3} \tag{50}$$

(equivalent to the differential equation that we have already integrated above), therefore

$$\frac{dR}{dt} = -\frac{2}{3} \frac{R^{5/2}}{\sqrt{GM}} \frac{96\mathcal{M}^{5/3}G^{5/3}}{5c^5} \omega^{11/3} = -\frac{64\mathcal{M}^{5/3}G^{7/6}}{5c^5\sqrt{M}} R^{5/2} \omega^{11/3}$$
(51)

or also

$$\frac{d}{dt}\left(R^{-3/2}\right) = \frac{96\mathcal{M}^{5/3}G^{7/6}}{5c^5\sqrt{M}}\omega^{11/3}$$
(52)

6.1 Maximum radiated power

The maximum radiated power for a system with generic masses is

$$P_{\max}(q) = \frac{32G^{7/3}}{5c^5} \frac{(m_1m_2)^2}{(m_1 + m_2)^{2/3}} \omega_0^{10/3}$$
(53)

$$= \frac{32G^{7/3}}{5c^5} \frac{(m_1m_2)^2}{(m_1+m_2)^{2/3}} \left[\frac{c^6}{8G^2} \frac{1}{(m_1+m_2)^2} \right]^{5/3}$$
(54)
$$= \frac{c^5}{5G} \frac{(m_1m_2)^2}{(m_1+m_2)^4} = \frac{c^5}{5G} \frac{q^2}{(1+m_1)^4} = \frac{c^5}{5G} \frac{q^2}{(1+m_1)^4}$$
(55)

$$=\frac{c^5}{5G}\frac{(m_1m_2)^2}{(m_1+m_2)^4} = \frac{c^5}{5G}\frac{q^2}{(1+q)^4} = \frac{c^5}{5G}\frac{q^2}{(1+q)^4}$$
(55)

where $q = m_2/m_1$.

The maximum is attained for q = 1:

$$P_{\max}(q=1) = \frac{c^5}{5G} \frac{1}{16}$$
(56)

so that

$$P_{\max}(q) = 16 \; \frac{q^2}{(1+q)^4} \; P_{\max}(q=1) \tag{57}$$

Notice that in this Newtonian approximation, the maximum radiated power does not depend on the total mass. On the contrary, the maximum frequency does depend on the total mass.

7 Maximum strain

Going back to equations (22) and (23), we see that the strain amplitude is

$$h_{\max} = \frac{4G(m_1r_1^2 + m_2r_2^2)\omega_0^2}{c^4D} = \frac{4G^{5/3}}{c^4D} \frac{m_1m_2}{(m_1 + m_2)^{1/3}} \omega_0^{2/3}$$
(58)

where ω_0 is the maximum orbital angular frequency (38). For equal masses $m_1 = m_2 = M/2$ and with the substitution

$$\omega_0^2 = (\pi f_0)^2 = \frac{c^6}{8G^2} \frac{1}{(m_1 + m_2)^2} = \frac{c^6}{8G^2} \frac{1}{M^2},$$
(59)

(see eq. 38), eq. (58) becomes

$$h_{\max} = \frac{G^{5/3}}{c^4 D} M^{5/3} (\pi f_0)^{2/3} = \frac{G^{5/3}}{c^4 D} M^{5/3} \left(\frac{c^6}{8G^2 M^2}\right)^{1/3} = \frac{GM}{2c^2 D} \\ = \frac{(GM_{\odot}/1 \text{ pc})(M/M_{\odot})}{2c^2 (D/1 \text{ pc})} \approx 2.4 \times 10^{-14} \frac{(M/M_{\odot})}{(D/1 \text{ pc})} \quad (60)$$

where 1 pc $\approx 3.0857 \times 10^{16}$ m. This means that a coalescence of two black holes with 1 solar mass each $(M = 2M_{\odot})$ at a distance approximately equal to that of the Virgo galaxy cluster (about 16.5 Mpc, see [2]), would produce a maximum strain approximately equal to 2.9×10^{-21} .

Remember that like all other estimates in this handout, this result is obtained in the context of the simplified Newtonian model.

Appendix: naive black hole calculations

The existence of black holes was predicted long before the advent of GR, from simple dynamical calculations.

In a gravitational field the *escape velocity* is defined as the threshold velocity that allows escape to infinity, so that the initial kinetic energy is equal to the gravitational potential energy at the starting radius:

$$\frac{1}{2}mv^2 = \frac{GMm}{R},\tag{61}$$

therefore

$$v = \sqrt{\frac{2GM}{R}}.$$
(62)

When the escape velocity equals the velocity of light, nothing can escape, not even light. This happens at the Schwarzschild radius R_S , such that

$$R_S = \frac{2GM}{c^2},\tag{63}$$

and rather surprisingly, this result coincides with the GR calculation for the Schwarzschild metric (non-rotating black hole).

When we express the mass in units of solar mass $(M_{\odot} \approx 2 \times 10^{30} \text{kg})$,

$$R_S = \frac{2GM_{\odot}(M/M_{\odot})}{c^2} \approx 3 \text{ km} \times (M/M_{\odot}), \tag{64}$$

so that a black hole with the mass of our Sun has a 3 km radius.

References

- Harsh Mathur, Katherine Brown, and Ashton Lowenstein. An analysis of the LIGO discovery based on introductory physics. *American Journal of Physics*, 85(9):676– 682, 2017.
- [2] Simona Mei, John P Blakeslee, Patrick Côté, John L Tonry, Michael J West, Laura Ferrarese, Andrés Jordán, Eric W Peng, André Anthony, and David Merritt. The ACS Virgo Cluster survey. XIII. SBF distance catalog and the three-dimensional structure of the Virgo Cluster. *The Astrophysical Journal*, 655(1):144, 2007.
- [3] LIGO Scientific, VIRGO collaborations, Benjamin P Abbott, R Abbott, TD Abbott, MR Abernathy, F Acernese, K Ackley, C Adams, T Adams, P Addesso, et al. The basic physics of the binary black hole merger GW150914. Annalen der Physik, 529(1-2):1600209, 2017.