The post-Newtonian approximation

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In this handout I follow the simplified explanation of the post-Newtonian approximation given by Clifford Will [1].

The approximation is based on the usual assumptions that gravitational fields are weak and that the characteristic motions of matter are slow compared with the speed of light. Then, we can characterize the system with a small parameter $\epsilon = (v/c)^2$. Since the $(v/c)^2$ ratio corresponds roughly to the ratio between the 00 component and one of the ii components of the stress-energy tensor, this parameter has the same order of magnitude as $p/\rho c^2$. Moreover, the parameter is also proportional to the kinetic energy divided by the mass M and by c^2; since we know from the virial theorem that the average kinetic energy of the system is proportional to the average of the potential energy, we find that the parameter has the same order as GM/rc^2. Putting it all together,

$$\epsilon \sim (v/c)^2 \sim GM/rc^2 \sim p/\rho c^2 \tag{1}$$

We found earlier that using the weak field and slow motion approximations, Einstein's equations can be linearized and that they become remarkably simple in the Lorentz gauge

$$\Box^2 \bar{h}^{\mu\nu} = \frac{16\pi G}{c^4} T^{\mu\nu} \tag{2}$$

$$\partial_\nu \bar{h}^{\mu\nu} = 0, \tag{3}$$

that a formal solution of this system is

$$\bar{h}^{\mu\nu}(ct, \mathbf{x}) = \frac{4G}{c^4} \int_{\text{source}} \frac{T^{\mu\nu}(ct - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}', \tag{4}$$

and that the stress-energy tensor satisfies the local conservation of energy

$$\partial_\nu T^{\mu\nu} = 0. \tag{5}$$

Since $T^{\mu\nu}$ encodes the distribution of mass-energy, eq. (5) can be used to infer the motion of the source from the solution (4). The pair of equations defines an iterative scheme, where we obtain a first iterate $h_0^{\mu\nu}$, we use it to determine the modified mass distribution and therefore the new $T^{\mu\nu}$. In turn, the new $T^{\mu\nu}$ is used to compute the next iterate $h_1^{\mu\nu}$, and so on.

Since we need both velocity and acceleration to obtain a solution of the equations of motion, we must iterate once more and obtain the iterate $h_1^{\mu\nu}$ to find the first order solution (first post-Newtonian order, 1PN, where the orders correspond to powers of ϵ). However, an evaluation of the power emitted as gravitational waves requires the evaluation of the third derivative of the mass distribution (of the reduced quadrupole tensor), and therefore we need one more iteration to evaluate the dissipation of energy. As an illustration, the two-body equation of motion becomes

$$\frac{d\mathbf{v}}{dt} = \frac{Gm}{r^2}\left(-\hat{\mathbf{n}} + \frac{1}{c^2}\mathbf{A}_{1PN} + \frac{1}{c^4}\mathbf{A}_{2PN} + \frac{1}{c^5}\mathbf{A}_{2.5PN} + \frac{1}{c^6}\mathbf{A}_{3PN} + \frac{1}{c^7}\mathbf{A}_{3.5PN} + \dots\right) \tag{6}$$

where $m = m_1 + m_2$, $\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$, $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$, and $\hat{\mathbf{n}} = (\mathbf{x}_1 - \mathbf{x}_2)/r$. The $\mathbf{A}$ terms have complex expressions, for example

$$\mathbf{A}_{1PN} = \left[(4+2\eta)\frac{Gm}{r} - (1+3\eta)v^2 + \frac{3}{2}\eta\dot{r}^2\right]\hat{\mathbf{n}} + (4-2\eta)\dot{r}\mathbf{v} \tag{7}$$

Quoting Clifford Will [1], *The post-Newtonian approximation has been remarkably effective as a tool for interpreting experimental tests of general relativity. This is because, in a broad class of alternative metric theories of gravity, it turns out that only the values of a set of numerical coefficients in the post-Newtonian expression for the spacetime metric vary from theory to theory. Thus one can encompass a wide range of alternative theories by simply introducing arbitrary parameters in place of the numerical coefficients. This idea dates back to Eddington in 1922, but the "parametrized post-Newtonian (PPN) framework" was fully developed by Nordtvedt and by Will in the period 1968–72. The framework contains 10 PPN parameters: γ, related to the amount of spatial curvature generated by mass; β, related to the degree of nonlinearity in the gravitational field; ξ, α_1, α_2, and α_3, which determine whether the theory predicts that local gravitational experiments could yield results that depend on the location or velocity of the reference frame; and ζ_1, ζ_2, ζ_3, and ζ_4, which describe whether the theory has appropriate momentum conservation laws. In general relativity, $\gamma = 1$, $\beta = 1$, and the remaining parameters all vanish.* This means that the PPN framework is very well adapted at expressing deviations from GR.

For more details, see [1].

References

[1] Clifford M Will. On the unreasonable effectiveness of the post-Newtonian approximation in gravitational physics. *Proceedings of the National Academy of Sciences*, 108(15):5938–5945, 2011.