Antenna patterns for a triple interferometer with triangular design

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Here I derive the antenna patterns for a triangular configuration like the one considered for the Einstein Telescope (see figure 1).

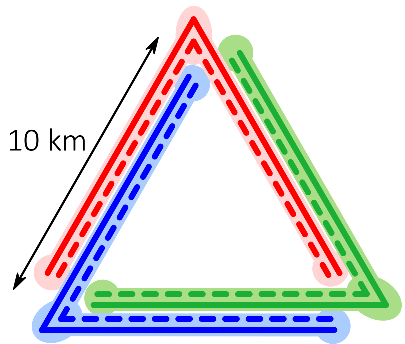

Figure 1: One of the possible configurations considered for the Einstein Telescope, an equilateral triangle. In this case there are three double interferometers with arms at 60°. Each interferometer is actually a pair of two instruments, one optimized for low frequencies, the other optimized for high frequencies. The quoted arm length is just an example of the alternatives being considered.

All the basic formulas are described in the "Antenna patterns" handout. Here, I use the following conventional unit vectors for the detector arms:

$$\mathbf{a} = \frac{\sqrt{3}}{2}\hat{\mathbf{e}}_x - \frac{1}{2}\hat{\mathbf{e}}_y; \quad \mathbf{b} = \frac{\sqrt{3}}{2}\hat{\mathbf{e}}_x + \frac{1}{2}\hat{\mathbf{e}}_y; \quad \mathbf{c} = \hat{\mathbf{e}}_y$$

The three interferometers are specified by the following unit vectors:

IFO 1: $\{\mathbf{a}, \mathbf{b}\}$

IFO 2: $\{\mathbf{c}, -\mathbf{a}\}$

IFO 3: $\{-\mathbf{b}, -\mathbf{c}\}$

Therefore, the detector tensors are

IFO 1:

$$\frac{1}{L}\left[d_{ij}^{(1)}\right] = \mathbf{a}\otimes\mathbf{a} - \mathbf{b}\otimes\mathbf{b} \tag{1}$$

$$= \left[\left(\frac{\sqrt{3}}{2}\hat{\mathbf{e}}_x - \frac{1}{2}\hat{\mathbf{e}}_y\right)\otimes\left(\frac{\sqrt{3}}{2}\hat{\mathbf{e}}_x - \frac{1}{2}\hat{\mathbf{e}}_y\right)\right] - \left[\left(\frac{\sqrt{3}}{2}\hat{\mathbf{e}}_x + \frac{1}{2}\hat{\mathbf{e}}_y\right)\otimes\left(\frac{\sqrt{3}}{2}\hat{\mathbf{e}}_x + \frac{1}{2}\hat{\mathbf{e}}_y\right)\right] \tag{2}$$

$$= \left(\frac{3}{4}\hat{\mathbf{e}}_x\otimes\hat{\mathbf{e}}_x - \frac{\sqrt{3}}{4}\hat{\mathbf{e}}_x\otimes\hat{\mathbf{e}}_y - \frac{\sqrt{3}}{4}\hat{\mathbf{e}}_y\otimes\hat{\mathbf{e}}_x + \frac{1}{4}\hat{\mathbf{e}}_y\otimes\hat{\mathbf{e}}_y\right)$$
$$- \left(\frac{3}{4}\hat{\mathbf{e}}_x\otimes\hat{\mathbf{e}}_x + \frac{\sqrt{3}}{4}\hat{\mathbf{e}}_x\otimes\hat{\mathbf{e}}_y + \frac{\sqrt{3}}{4}\hat{\mathbf{e}}_y\otimes\hat{\mathbf{e}}_x + \frac{1}{4}\hat{\mathbf{e}}_y\otimes\hat{\mathbf{e}}_y\right) \tag{3}$$

IFO 2:

$$\frac{1}{L}\left[d_{ij}^{(2)}\right] = \mathbf{c}\otimes\mathbf{c} - \mathbf{a}\otimes\mathbf{a} = \tag{4}$$

$$= (\hat{\mathbf{e}}_y\otimes\hat{\mathbf{e}}_y) - \left(\frac{3}{4}\hat{\mathbf{e}}_x\otimes\hat{\mathbf{e}}_x - \frac{\sqrt{3}}{4}\hat{\mathbf{e}}_x\otimes\hat{\mathbf{e}}_y - \frac{\sqrt{3}}{4}\hat{\mathbf{e}}_y\otimes\hat{\mathbf{e}}_x + \frac{1}{4}\hat{\mathbf{e}}_y\otimes\hat{\mathbf{e}}_y\right) \tag{5}$$

IFO 3:

$$\frac{1}{L}\left[d_{ij}^{(3)}\right] = \mathbf{b}\otimes\mathbf{b} - \mathbf{c}\otimes\mathbf{c} \tag{6}$$

$$= \left(\frac{3}{4}\hat{\mathbf{e}}_x\otimes\hat{\mathbf{e}}_x + \frac{\sqrt{3}}{4}\hat{\mathbf{e}}_x\otimes\hat{\mathbf{e}}_y + \frac{\sqrt{3}}{4}\hat{\mathbf{e}}_y\otimes\hat{\mathbf{e}}_x + \frac{1}{4}\hat{\mathbf{e}}_y\otimes\hat{\mathbf{e}}_y\right) - (\hat{\mathbf{e}}_y\otimes\hat{\mathbf{e}}_y) \tag{7}$$

Assuming the long wavelength approximations (i.e., no frequency dependence of the antenna patterns), the expressions above can be simplified to obtain the following useful forms:

IFO 1:

$$\frac{1}{L}\left[d_{ij}^{(1)}\right] = -\frac{\sqrt{3}}{2}\hat{\mathbf{e}}_x\otimes\hat{\mathbf{e}}_y - \frac{\sqrt{3}}{2}\hat{\mathbf{e}}_y\otimes\hat{\mathbf{e}}_x = \begin{pmatrix} 0 & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{8}$$

IFO 2:

$$\frac{1}{L}\left[d_{ij}^{(2)}\right] = -\frac{3}{4}\hat{\mathbf{e}}_x\otimes\hat{\mathbf{e}}_x + \frac{\sqrt{3}}{4}\hat{\mathbf{e}}_x\otimes\hat{\mathbf{e}}_y + \frac{\sqrt{3}}{4}\hat{\mathbf{e}}_y\otimes\hat{\mathbf{e}}_x + \frac{3}{4}\hat{\mathbf{e}}_y\otimes\hat{\mathbf{e}}_y = \begin{pmatrix} -\frac{3}{4} & \frac{\sqrt{3}}{4} & 0 \\ \frac{\sqrt{3}}{4} & \frac{3}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{9}$$

IFO 3:

$$\frac{1}{L}\left[d_{ij}^{(3)}\right] = \frac{3}{4}\hat{\mathbf{e}}_x\otimes\hat{\mathbf{e}}_x + \frac{\sqrt{3}}{4}\hat{\mathbf{e}}_x\otimes\hat{\mathbf{e}}_y + \frac{\sqrt{3}}{4}\hat{\mathbf{e}}_y\otimes\hat{\mathbf{e}}_x - \frac{3}{4}\hat{\mathbf{e}}_y\otimes\hat{\mathbf{e}}_y = \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & 0 \\ \frac{\sqrt{3}}{4} & -\frac{3}{4} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{10}$$

Now, recalling the polarization tensors for a source in direction (θ, ϕ)

$$
\begin{aligned}
\left[\epsilon_+^{ij}\right] &= \left[(\hat{e}_x^R)^i (\hat{e}_x^R)^j\right] - \left[(\hat{e}_y^R)^i (\hat{e}_y^R)^j\right] \\
&= \begin{pmatrix}
\cos^2\theta\cos^2\phi - \sin^2\phi & \cos^2\theta\sin\phi\cos\phi + \sin\phi\cos\phi & -\sin\theta\cos\theta\cos\phi \\
\cos^2\theta\sin\phi\cos\phi + \sin\phi\cos\phi & \cos^2\theta\sin^2\phi - \cos^2\phi & -\sin\theta\cos\theta\sin\phi \\
-\sin\theta\cos\theta\cos\phi & -\sin\theta\cos\theta\sin\phi & \sin^2\theta
\end{pmatrix}
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\left[\epsilon_\times^{ij}\right] &= \left[(\hat{e}_x^R)^i (\hat{e}_y^R)^j\right] + \left[(\hat{e}_y^R)^i (\hat{e}_x^R)^j\right] \\
&= \begin{pmatrix}
-\cos\theta\sin 2\phi & \cos\theta\cos^2\phi - \cos\theta\sin^2\phi & \sin\theta\sin\phi \\
\cos\theta\cos^2\phi - \cos\theta\sin^2\phi & \cos\theta\sin 2\phi & -\sin\theta\cos\phi \\
\sin\theta\sin\phi & -\sin\theta\cos\phi & 0
\end{pmatrix},
\end{aligned}
\tag{12}
$$

we find

$$F_+^{(1)}(\theta,\phi) = \frac{\sqrt{3}}{2}\left[\frac{1}{2}\left(\cos^2\theta + 1\right)\sin 2\phi\right] \tag{13a}$$

$$F_\times^{(1)}(\theta,\phi) = \frac{\sqrt{3}}{2}\left[\cos\theta\cos 2\phi\right] \tag{13b}$$

for the first interferometer, and similar expressions for the other two interferometers in the triangular design.

When we include the polarization angle as well, we find

$$F_+^{(1)}(\theta,\phi) = \frac{\sqrt{3}}{2}\left[\frac{1}{2}\left(\cos^2\theta + 1\right)\sin 2\phi\cos 2\psi - \cos\theta\cos 2\phi\sin 2\psi\right] \tag{14a}$$

$$F_\times^{(1)}(\theta,\phi) = \frac{\sqrt{3}}{2}\left[\frac{1}{2}\left(\cos^2\theta + 1\right)\sin 2\phi\sin 2\psi + \cos\theta\cos 2\phi\cos 2\psi\right] \tag{14b}$$