Antenna patterns

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In this handout we derive the formula for the antenna patterns of a Michelson-type gravitational-wave detector. After a general introduction, we obtain the *three-term formula*, using an argument adapted from B. Schutz [3]. We carry out all calculations in detail and finally we obtain the general formula for the antenna patterns in the low–frequency approximation.

1 Introduction

The antenna patterns define the sensitivity of a gravitational wave antenna to incoming gravitational waves. It is useful to recall the similar situation for electromagnetic waves. In the case of a simple dipole antenna, where electrons are accelerated by incoming EM waves along the axis of the antenna, it is easy to show that the intensity (power per unit area per unit time) is given by the norm of the Poynting vector

$$|\mathbf{S}(\mathbf{r},t)| = \frac{e^2}{16\pi^2\varepsilon_0 c^2}\,\frac{a^2\sin^2\theta}{r^2} \tag{1}$$

where a is the norm of the acceleration, which is directed along the z axis. This intensity can be represented as in figure 1 which shows the electromagnetic radiation pattern of a dipole antenna.

2 The three-term formula

We consider a pulse of light traveling between a freely-falling light source and a freely-falling mirror. The segment joining the two objects defines the x-axis, while a GW source lies on the z-axis. The generic null line interval is

$$ds^2 = c^2dt^2 - [1 + h_+(t - z/c)]dx^2 - [1 - h_+(t - z/c)]dy^2 - dz^2 = 0, \tag{2}$$

and since the light pulse moves only along the x-axis

$$ds^2 = c^2dt^2 - [1 + h_+(t - z/c)]dx^2 = 0. \tag{3}$$

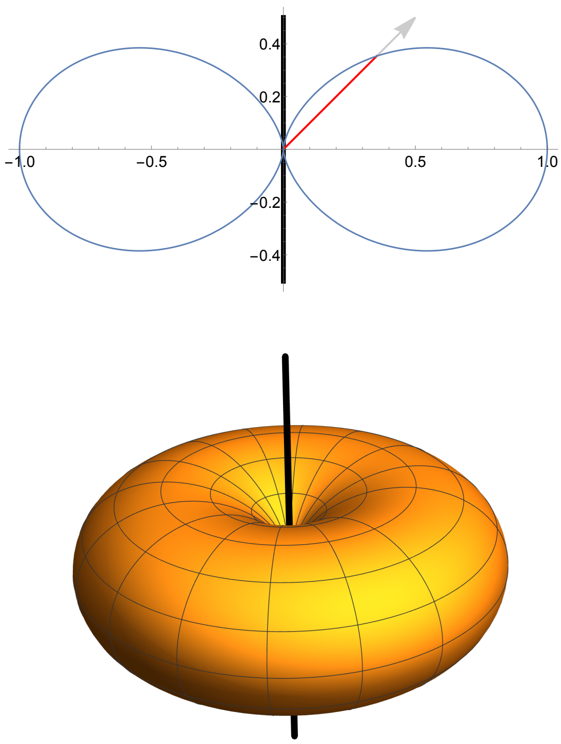

Figure 1: Electromagnetic radiation diagram, the antenna pattern of a simple dipole antenna. The antenna points in the z direction, and this is also the direction of the acceleration vector of the electrons in the metal wire. Upper panel: cross–section of the antenna pattern along the xz plane, the gray arrow represents a specific direction and the red segment is proportional to the intensity in that direction. Lower panel: radiation diagram in a 3D representation.

Next, we note that there is no spatial phase change in the interferometer plane because $z =$ constant (which we neglect) and we find

$$c^2dt^2 = [1 + h_+(t)]dx^2, \tag{4}$$

and therefore

$$dt = \frac{1}{c}\sqrt{1 + h_+(t)}dx \approx \frac{1}{c}\left[1 + \frac{1}{2}h_+(t)\right]dx. \tag{5}$$

We separate variables as follows

$$dx = \frac{cdt}{1 + \frac{1}{2}h_+(t)} \approx cdt\left(1 - \frac{1}{2}h_+(t)\right), \tag{6}$$

and integrate over the whole length L_x of the segment joining source and mirror

$$L_x = c \int_{t_0}^{t_1} \left(1 - \frac{1}{2} h_+(t)\right) dt = c(t_1 - t_0) - \frac{c}{2} \int_{t_0}^{t_1} h_+(t) dt, \tag{7}$$

therefore

$$\begin{aligned} t_1 \approx t_0 + \frac{L_x}{c} + \frac{1}{2} \int_{t_0}^{t_1} h_+(t) dt = t_0 + \frac{L_x}{c} + \frac{1}{2} \int_{t_0}^{t_0 + L_x/c + O(h_+)} & h_+(t) dt \\ \approx t_0 + \frac{L_x}{c} + \frac{1}{2c} \int_0^{L_x} & h_+(t_0 + x/c) dx, \end{aligned} \tag{8}$$

where the last approximation holds because the integral is already of order h_+. In the last step, we also switched to x as integration variable. Then, taking into account the backward path from mirror to source, we evaluate the return time t_2

$$t_2 = t_0 + \frac{2L_x}{c} + \frac{1}{2c} \int_0^{L_x} h_+(t_0 + x/c) dx + \frac{1}{2c} \int_0^{L_x} h_+(t_0 + L_x/c + x/c) dx. \tag{9}$$

Taking the derivative with respect to the start time t_0, we find

$$\frac{dt_2}{dt_0} = 1 + \frac{1}{2c} \int_0^{L_x} h'_+(t_0 + x/c) dx + \frac{1}{2c} \int_0^{L_x} h'_+(t_0 + L_x/c + x/c) dx \tag{10}$$

$$= 1 + \frac{1}{2} \left[h_+(t_0 + L_x/c) - h_+(t_0)\right] + \frac{1}{2} \left[h_+(t_0 + 2L_x/c) - h_+(t_0 + L_x/c)\right] \tag{11}$$

$$= 1 + \frac{1}{2} \left[h_+(t_0 + 2L_x/c) - h_+(t_0)\right] \tag{12}$$

The last result holds for a plane gravitational wave with a wave vector parallel to the z-axis. **Notice also that the intermediate value depending on $t_0 + L_x/c$ cancels out**.

Next, we generalize this result to a rotated system, where the gravitational wave impinges on the source–mirror system with an angle θ with respect to the z-axis, as in figure 2. The rotation of a **covariant vector** about the y-axis is represented by the (space) rotation matrix

$$R_i^j = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{13}$$

therefore the space part of the strain of a + polarized GW (remember that we are using the TT gauge!)

$$h_{ij} = \begin{pmatrix} h_+ & 0 & 0 \\ 0 & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14}$$

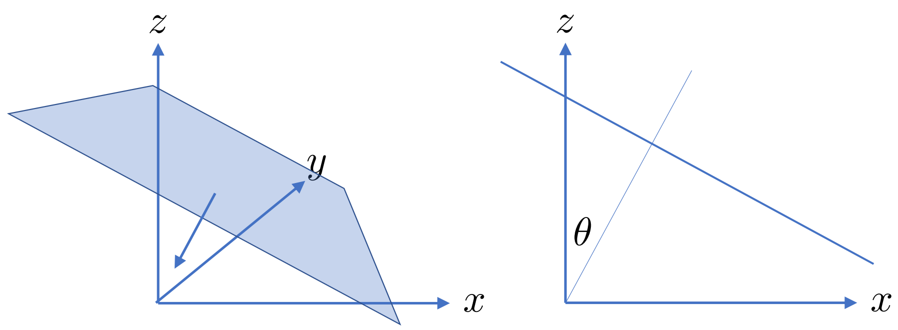

Figure 2: Reference frame for the treatment of an incoming gravitational wave with angle θ with respect to the z-axis. The left panel shows a perspective view. The right panel shows a view along the y-axis: the reference frame is rotated about y with respect to that with direction of the wave along the z-axis. In the xz plane, the wave vector of the gravitational wave is antiparallel to the unit vector $(\sin\theta, \cos\theta)$ and wavefronts of constant phase are identified by the equation $x\sin\theta + z\cos\theta = \text{const}$.

transforms to

$$h_{ij}^{\text{rot}} = R_i^m R_j^n h_{mn}, \tag{15}$$

in particular, the strain along the x-axis transforms to

$$h_{xx}^{\text{rot}} = R_x^m R_x^n h_{mn} = h_+ \cos^2\theta. \tag{16}$$

In the rotated frame the wavefronts of constant phase are identified by the equation

$$x\sin\theta + z\cos\theta = \text{const} \tag{17}$$

(see Fig. 2), therefore the spatial part of the phase change measured along the x axis (i.e., at $z = 0$) is just $x\sin\theta/c$ so that the total contribution to phase change is $x(1 \pm \sin\theta)/c$ where the x term corresponds to the time phase change translated into space coordinate and the $-\sin\theta$ term corresponds to the actual space contribution, with a $-$ sign that corresponds to the forward path. Taking into account both changes, we modify eq. (8) as follows

$$t_1 = t_0 + \frac{L_x}{c} + \frac{\cos^2\theta}{2c}\int_0^{L_x} h_+[t_0 + x(1-\sin\theta)/c]dx \tag{18}$$

The return leg is similar, with the difference that the spatial phase must be counted backwards $-(L_x - x)\sin\theta/c$, and we find

$$\begin{aligned} t_2 = t_0 + \frac{2L_x}{c} + \frac{\cos^2\theta}{2c}\int_0^{L_x} h_+[t_0 + x(1-\sin\theta)/c]dx \\ + \frac{\cos^2\theta}{2c}\int_0^{L_x} h_+[t_0 + L_x(1-\sin\theta)/c + x(1+\sin\theta)/c]dx \end{aligned} \tag{19}$$

Taking once again the derivative with respect to the start time t_0, we find

$$\begin{aligned}\frac{dt_2}{dt_0} &= 1+\frac{\cos^2\theta}{2c}\int_0^{L_x} h'_+[t_0+x(1-\sin\theta)/c]dx \\ &+\frac{\cos^2\theta}{2c}\int_0^{L_x} h'_+[t_0+L_x(1-\sin\theta)/c+x(1+\sin\theta)/c]dx \end{aligned}\tag{20}$$

$$\begin{aligned}&= 1+\frac{\cos^2\theta}{2}\left\{\frac{h_+[t_0+L_x(1-\sin\theta)/c]-h_+(t_0)}{1-\sin\theta}\right.\\ &\quad\left.+\frac{h_+[t_0+L_x(1-\sin\theta)/c+L_x(1+\sin\theta)/c]-h_+[t_0+L_x(1-\sin\theta)/c]}{1+\sin\theta}\right\}\end{aligned}\tag{21}$$

$$\begin{aligned}&= 1+\frac{1}{2}\{(1+\sin\theta)h_+[t_0+L_x(1-\sin\theta)/c]-(1+\sin\theta)h_+(t_0)\\ &\quad+(1-\sin\theta)h_+[t_0+2L_x/c]-(1-\sin\theta)h_+[t_0+L_x(1-\sin\theta)/c]\}\end{aligned}\tag{22}$$

$$= 1+\frac{1}{2}\left\{(1-\sin\theta)\ h_+\left[t_0+2\frac{L_x}{c}\right]-(1+\sin\theta)\ h_+(t_0)+2\sin\theta\ h_+\left[t_0+\frac{L_x}{c}(1-\sin\theta)\right]\right\}\tag{23}$$

the last line (23) is the *three-term formula*.

3 The antenna patterns

First, we expand the three-term formula for small L_x, i.e., assuming that $L_x \ll \lambda_{\rm GW}$ (the arm length is much smaller than the wavelength of the gravitational wave)

$$\frac{dt_2}{dt_0} = 1+\frac{1}{2}\left\{(1-\sin\theta)\ h_+\left[t_0+2\frac{L_x}{c}\right]-(1+\sin\theta)\ h_+(t_0)+2\sin\theta\ h_+\left[t_0+\frac{L_x}{c}(1-\sin\theta)\right]\right\}\tag{24}$$

$$\begin{aligned}&\approx 1+\frac{1}{2}\left\{(1-\sin\theta)\left[h_+(t_0)+\frac{2L_x}{c}\dot h_+(t_0)\right]-(1+\sin\theta)\ h_+(t_0)\right.\\ &\quad\left.+2\sin\theta\ \left[h_+(t_0)+(1-\sin\theta)\frac{L_x}{c}\dot h_+(t_0)\right]\right\}\end{aligned}\tag{25}$$

$$= 1+\frac{L_x}{c}\cos^2\theta\dot h_+(t_0)\tag{26}$$

and we note that the scalar expression contains $\dot h_+(t_0)$ which comes from a rank 2 tensor[1]. The only available tensor objects are the derivative of the gravitational wave

[1]The expression is similar to the one quoted in [2], with the substitution sin $\to$ cos. The reason is that the angle in [2] has an additional 90° rotation.

tensor $\dot{h}_{ij}$ and the unit vector that specifies the direction of motion of the light pulses (the x axis). This means that the scalar must have the following form in terms of tensor quantities

$$\left.\frac{dt_2}{dt_0}\right|_{\mathrm{x-arm}} = 1 + \frac{L_x}{c}\dot{h}_{ij}\hat{e}_x^i\hat{e}_x^j, \tag{27}$$

where $\hat{e}_x$ is the unit vector in the direction of the x arm[2], and similarly for the y arm of an interferometer with x and y arms

$$\left.\frac{dt_2}{dt_0}\right|_{\mathrm{y-arm}} = 1 + \frac{L_y}{c}\dot{h}_{ij}\hat{e}_y^i\hat{e}_y^j, \tag{28}$$

so that the global response of the interferometers is (with $L_x = L_x = L$)

$$\frac{d\delta t}{dt_0} = \left.\frac{dt_2}{dt_0}\right|_{\mathrm{x-arm}} - \left.\frac{dt_2}{dt_0}\right|_{\mathrm{y-arm}} = \frac{L}{c}\dot{h}_{ij}\left(\hat{e}_x^i\hat{e}_x^j - \hat{e}_y^i\hat{e}_y^j\right) \tag{29}$$

Finally, integrating the last equation, we find the **differential return time**

$$\delta t = \frac{L}{c}h_{ij}\left(\hat{e}_x^i\hat{e}_x^j - \hat{e}_y^i\hat{e}_y^j\right) \tag{30}$$

Expression (30) can be recast in the simpler form

$$\delta t = \frac{1}{c}h_{ij}d_{ij} \tag{31}$$

where $\mathbf{d}$ is the *detector tensor* with components

$$d_{ij} = L\left(\hat{e}_x^i\hat{e}_x^j - \hat{e}_y^i\hat{e}_y^j\right). \tag{32}$$

This expression can be cast in the even more compact form

$$\delta t = \frac{1}{c}\mathbf{h}:\mathbf{d} \tag{33}$$

if we use the shorthand notation $\mathbf{h}:\mathbf{d} \equiv h_{ij}d_{ij}$.

Note that the definition of the detector tensor is similar to that of the polarization tensors that we defined earlier

$$\mathbf{e}_+ = \hat{e}_x^i\hat{e}_x^j - \hat{e}_y^i\hat{e}_y^j, \quad \mathbf{e}_\times = \hat{e}_x^i\hat{e}_y^j + \hat{e}_y^i\hat{e}_x^j, \tag{34}$$

although their two expressions are usually computed in different frames of reference.

Turning now to the **differential length change**, this is half the length determined by the differential return time, and eq. (33) becomes

$$\delta L = \frac{1}{2}\mathbf{h}:\mathbf{d} \tag{35}$$

[2]Indeed, the rotated $\hat{e}_x$ unit vector has the representation $(\cos\theta, 0, -\sin\theta)$ in the rotated system.

Figure 3 shows the geometry of the situation we are describing. In this case the polarization tensors in the TT frame are obtained from the vectors labeled $\hat{e}_x^R$, etc., shown in the left panel. However, this is a very special choice, where the $\hat{e}_x^R$ vector is parallel the x-axis of the detector frame before the rotation that brings it into the proper sky direction. In general, the system in the TT frame is also rotated by an angle ψ, as shown in the right panel of figure 3 (see also figure 4 for another view of the rotation angles).

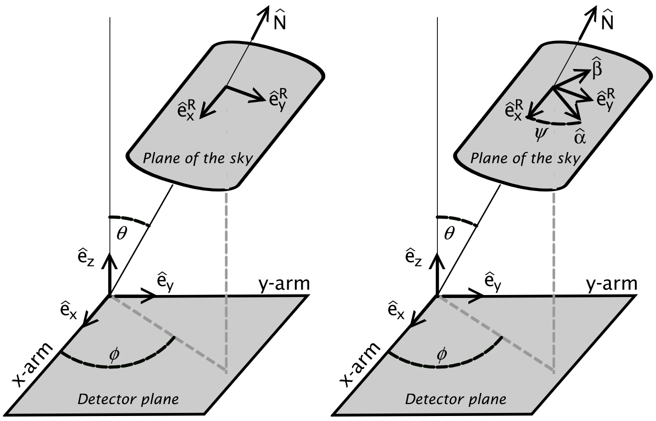

Figure 3: The relative orientation of the sky and detector frames (left panel) and the effect of a rotation by the angle ψ in the sky frame (right panel), from Sathyaprakash and Schutz [2]

With this additional rotation, the polarization tensors are defined by

$$\epsilon_+ = \hat{\alpha}^i\hat{\alpha}^j - \hat{\beta}^i\hat{\beta}^j, \quad \epsilon_\times = \hat{\alpha}^i\hat{\beta}^j + \hat{\beta}^i\hat{\alpha}^j, \tag{36}$$

and since the effect of the rotation is

$$\hat{\alpha} = \hat{e}_x^R \cos\psi + \hat{e}_y^R \sin\psi \tag{37}$$

$$\hat{\beta} = -\hat{e}_x^R \sin\psi + \hat{e}_y^R \cos\psi \tag{38}$$

we find

$$\begin{aligned}
\epsilon_+^{ij} &= \hat{\alpha}^i\hat{\alpha}^j - \hat{\beta}^i\hat{\beta}^j && (39)\\
&= \left[(\hat{e}_x^R)^i \cos\psi + (\hat{e}_y^R)^i \sin\psi\right]\left[(\hat{e}_x^R)^j \cos\psi + (\hat{e}_y^R)^j \sin\psi\right] \\
&\qquad - \left[-(\hat{e}_x^R)^i_x \sin\psi + (\hat{e}_y^R)^i \cos\psi\right]\left[-(\hat{e}_x^R)^j \sin\psi + (\hat{e}_y^R)^j \cos\psi\right] && (40)\\
&= (\hat{e}_x^R)^i(\hat{e}_x^R)^j \cos 2\psi + (\hat{e}_x^R)^i(\hat{e}_y^R)^j \sin 2\psi + (\hat{e}_y^R)^i(\hat{e}_x^R)^j \sin 2\psi - (\hat{e}_y^R)^i(\hat{e}_y^R)^j \cos 2\psi && (41)\\
&= \mathbf{e}_+^{ij} \cos 2\psi + \mathbf{e}_\times^{ij} \sin 2\psi && (42)
\end{aligned}$$

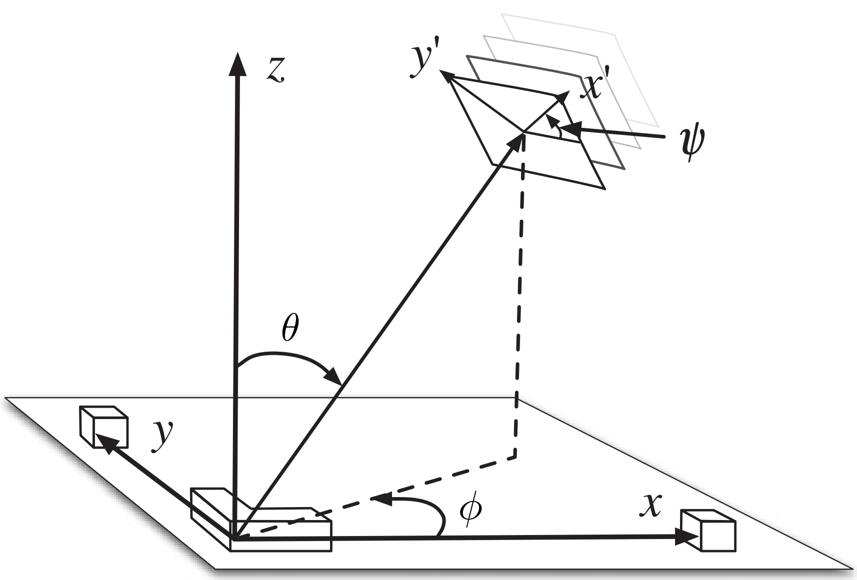

Figure 4: A representation of the individual rotations used to obtain the expression for the antenna patterns, from [1].

with a similar result for $\epsilon_\times^{ij}$:

$$\epsilon_+ = \mathbf{e}_+ \cos 2\psi + \mathbf{e}_\times \sin 2\psi \tag{43a}$$
$$\epsilon_\times = -\mathbf{e}_+ \sin 2\psi + \mathbf{e}_\times \cos 2\psi \tag{43b}$$

Spelling out the differential length change (35), we see that it is a function of the angles θ, ϕ, and ψ, i.e.,

$$\frac{\delta L}{L} = F_+(\theta, \phi, \psi)\; h_+(t) + F_\times(\theta, \phi, \psi)\; h_\times(t) \tag{44}$$

where the coefficients F_+ and $F_\times$ are the *antenna patterns* of the interferometer. The antenna patterns are obtained from the rotated basis vectors. The detailed calculations follow:

- rotation matrix

$$R = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\phi & -\sin\phi & \sin\theta\cos\phi \\ \cos\theta\sin\phi & \cos\phi & \sin\theta\sin\phi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \tag{45}$$

- representation of the rotated xy basis vectors

$$\left[(\hat{e}_x^R)^i\right] = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}; \quad \left[(\hat{e}_y^R)^i\right] = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \tag{46}$$

- partial tensors

$$\left[(\hat{e}_x^R)^i(\hat{e}_x^R)^j\right] = \begin{pmatrix} \cos^2\theta\cos^2\phi & \cos^2\theta\sin\phi\cos\phi & -\sin\theta\cos\theta\cos\phi \\ \cos^2\theta\sin\phi\cos\phi & \cos^2\theta\sin^2\phi & -\sin\theta\cos\theta\sin\phi \\ -\sin\theta\cos\theta\cos\phi & -\sin\theta\cos\theta\sin\phi & \sin^2\theta \end{pmatrix} \tag{47a}$$

$$\left[(\hat{e}_y^R)^i(\hat{e}_y^R)^j\right] = \begin{pmatrix} \sin^2\phi & -\sin\phi\cos\phi & 0 \\ -\sin\phi\cos\phi & \cos^2\phi & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{47b}$$

$$\left[(\hat{e}_x^R)^i(\hat{e}_y^R)^j\right] = \begin{pmatrix} -\cos\theta\cos\phi\sin\phi & \cos\theta\cos^2\phi & 0 \\ -\cos\theta\sin^2\phi & \cos\theta\sin\phi\cos\phi & 0 \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & 0 \end{pmatrix} \tag{47c}$$

$$\left[(\hat{e}_y^R)^i(\hat{e}_x^R)^j\right] = \begin{pmatrix} -\cos\theta\cos\phi\sin\phi & -\cos\theta\sin^2\phi & \sin\theta\sin\phi \\ \cos\theta\cos^2\phi & \cos\theta\sin\phi\cos\phi & -\sin\theta\cos\phi \\ 0 & 0 & 0 \end{pmatrix} \tag{47d}$$

- polarization tensors

$$\begin{aligned}\left[\epsilon_+^{ij}\right] &= \left[(\hat{e}_x^R)^i(\hat{e}_x^R)^j\right] - \left[(\hat{e}_y^R)^i(\hat{e}_y^R)^j\right] \\ &= \begin{pmatrix} \cos^2\theta\cos^2\phi - \sin^2\phi & \cos^2\theta\sin\phi\cos\phi + \sin\phi\cos\phi & -\sin\theta\cos\theta\cos\phi \\ \cos^2\theta\sin\phi\cos\phi + \sin\phi\cos\phi & \cos^2\theta\sin^2\phi - \cos^2\phi & -\sin\theta\cos\theta\sin\phi \\ -\sin\theta\cos\theta\cos\phi & -\sin\theta\cos\theta\sin\phi & \sin^2\theta \end{pmatrix}\end{aligned} \tag{48}$$

$$\begin{aligned}\left[\epsilon_\times^{ij}\right] &= \left[(\hat{e}_x^R)^i(\hat{e}_y^R)^j\right] + \left[(\hat{e}_y^R)^i(\hat{e}_x^R)^j\right] \\ &= \begin{pmatrix} -\cos\theta\sin 2\phi & \cos\theta\cos^2\phi - \cos\theta\sin^2\phi & \sin\theta\sin\phi \\ \cos\theta\cos^2\phi - \cos\theta\sin^2\phi & \cos\theta\sin 2\phi & -\sin\theta\cos\phi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & 0 \end{pmatrix}\end{aligned} \tag{49}$$

- the detector tensor

$$\left[d^{ij}\right] = L\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{50}$$

- detector response

$$\frac{\delta L}{L} = \frac{1}{2}h_{ij}d^{ij} = \frac{1}{2}h_+\left(\cos^2\theta + 1\right)\cos 2\phi - h_\times\cos\theta\sin 2\phi \tag{51}$$

- detector response including the polarization angle in the source reference frame, Eq. (43)

$$\frac{\delta L}{L} = \frac{1}{2L} h_{ij} d^{ij} = h_+ \left[\frac{1}{2} \left(\cos^2\theta + 1 \right) \cos 2\phi \cos 2\psi - \cos\theta \sin 2\phi \sin 2\psi \right] - h_\times \left[\frac{1}{2} \left(\cos^2\theta + 1 \right) \cos 2\phi \sin 2\psi + \cos\theta \sin 2\phi \cos 2\psi \right] \quad (52)$$

- antenna patterns

$$F_+ = \frac{1}{2}(1 + \cos^2\theta) \cos 2\phi \cos 2\psi - \cos\theta \sin 2\phi \sin 2\psi \quad (53)$$

$$F_\times = \frac{1}{2}(1 + \cos^2\theta) \cos 2\phi \sin 2\psi + \cos\theta \sin 2\phi \cos 2\psi \quad (54)$$

Figure 5 shows a graphical representation of F_+ and $F_\times$.

- formal definition of the antenna patterns

$$\frac{\delta L}{L} = \frac{1}{2L} h_{ij} d^{ij} = \frac{1}{2} \left(h_+ \epsilon_+^{ij} + h_\times \epsilon_\times^{ij} \right) \left(\hat{e}_x^i \hat{e}_x^j - \hat{e}_y^i \hat{e}_y^j \right) \quad (55)$$

Therefore

$$F_A = \frac{1}{2} \epsilon_A^{ij} \left(\hat{e}_x^i \hat{e}_x^j - \hat{e}_y^i \hat{e}_y^j \right) \quad (56)$$

where A is the polarization state $+$ or $\times$.

References

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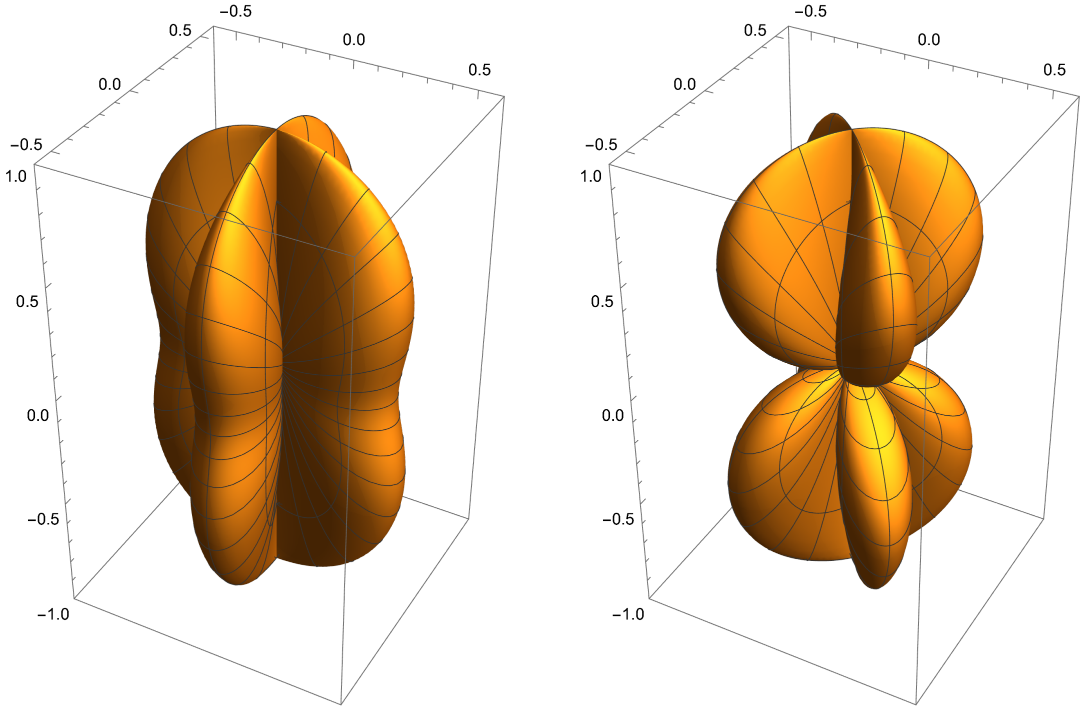

Figure 5: Antenna patterns F_+ (left) and $F_\times$ (right).