Antenna patterns

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In this handout I derive the formula for the antenna patterns of a Michelson-type gravitational-wave detector for a pure + polarization. Initially, we derive the *three-term formula*, using an argument adapted from B. Schutz [3].

The three-term formula

We consider a pulse of light traveling between a freely-falling light source and a freely-falling mirror. The segment joining the two objects defines the x-axis, while a GW source lies on the z-axis. The null line interval is

$$ds^2 = c^2 dt^2 - [1 + h_+(t - z/c)]dx^2 - [1 - h_+(t - z/c)]dy^2 - dz^2 = 0, \tag{1}$$

which means that with the light moving only along the x-axis, and neglecting the spatial phase change (this means that the wavelength of the GW is much longer than the distance L between source and mirror) we find

$$c^2 dt^2 = [1 + h_+(t)]dx^2, \tag{2}$$

and therefore

$$dt = \frac{1}{c}\sqrt{1 + h_+(t)}dx \approx \frac{1}{c}\left[1 + \frac{1}{2}h_+(t)\right] dx. \tag{3}$$

We can separate variables as follows

$$dx = \frac{cdt}{1 + \frac{1}{2}h_+(t)} \approx cdt\left(1 - \frac{1}{2}h_+(t)\right) \tag{4}$$

and integrate over the whole length

$$L_x = c\int_{t_0}^{t_1}\left(1 - \frac{1}{2}h_+(t)\right) dt = c(t_1 - t_0) - \frac{c}{2}\int_{t_0}^{t_1} h_+(t)dt. \tag{5}$$

Therefore

$$\begin{aligned} t_1 &\approx t_0 + \frac{L_x}{c} + \frac{1}{2}\int_{t_0}^{t_1} h_+(t)dt \approx t_0 + \frac{L_x}{c} + \frac{1}{2}\int_{t_0}^{t_0+L_x/c} h_+(t)dt \\ &= t_0 + \frac{L_x}{c} + \frac{1}{2c}\int_0^{L_x} h_+(t_0 + x/c)dx \end{aligned} \tag{6}$$

because the integral is already of order h. Considering also the backward path from mirror to source, the return time t_2 is

$$t_2 = t_0 + \frac{2L_x}{c} + \frac{1}{2c}\int_0^{L_x} h_+(t_0 + x/c)dx + \frac{1}{2c}\int_0^{L_x} h_+(t_0 + L/c + x/c)dx. \quad (7)$$

Taking the derivative with respect to the start time t_0, we find

$$\frac{dt_2}{dt_0} = 1 + \frac{1}{2c}\int_0^{L_x} h'_+(t_0 + x/c)dx + \frac{1}{2c}\int_0^{L_x} h'_+(t_0 + L_x/c + x/c)dx \quad (8)$$

$$= 1 + \frac{1}{2}\left[h_+(t_0 + L_x/c) - h_+(t_0)\right] + \frac{1}{2}\left[h_+(t_0 + 2L_x/c) - h_+(t_0 + L_x/c)\right] \quad (9)$$

$$= 1 + \frac{1}{2}\left[h_+(t_0 + 2L_x/c) - h_+(t_0)\right] \quad (10)$$

The last result holds for a plane gravitational wave with a wave vector parallel to the z-axis. Next, we generalize this result to a rotated system, where the gravitational wave impinges on the source–mirror system with an angle θ with respect to the z-axis, as in figure 1. The rotation of a covariant vector about the y-axis is represented by the

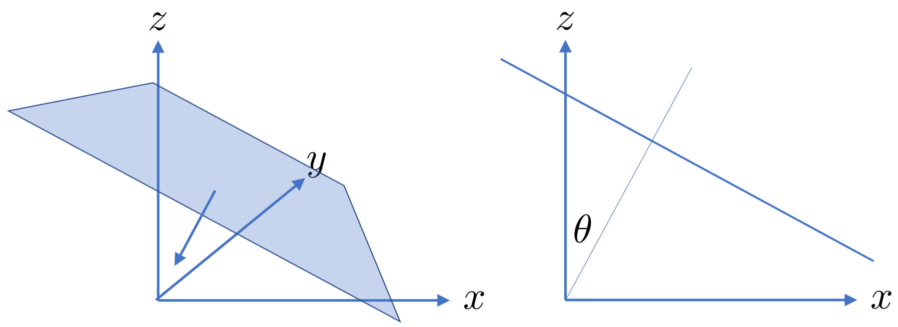

Figure 1: Reference frame for the treatment of an incoming gravitational wave with angle θ with respect to the z-axis. The left panel shows a perspective view. The right panel shows a view along the y-axis: the reference frame is rotated about y with respect to that with direction of the wave along the z-axis.

(space) rotation matrix

$$R_i^j = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad (11)$$

therefore the space part of the strain of a + polarized GW (TT gauge!)

$$h_{ij} = \begin{pmatrix} h_+ & 0 & 0 \\ 0 & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (12)$$

transforms to

$$h_{ij}^{\text{rot}} = R_i^m R_j^n h_{mn}, \tag{13}$$

in particular, the strain along the x-axis transforms to

$$h_{xx}^{\text{rot}} = R_x^m R_x^n h_{mn} = h_+ \cos^2\theta. \tag{14}$$

In the rotated frame, the spatial part of the phase change of the gravitational wave is

$$(z\cos\theta - x\sin\theta)/c \tag{15}$$

so that the spatial phase change measured along the x axis (i.e., at $z = 0$) is just $x\sin\theta/c$, and we can modify eq. (6) as follows

$$t_1 = t_0 + \frac{L_x}{c} + \frac{\cos^2\theta}{2c}\int_0^{L_x} h_+[t_0 + x(1-\sin\theta)/c]dx \tag{16}$$

The return leg is similar, with the difference that the spatial phase must be counted backwards $-(L-x)\sin\theta/c$, and we find

$$\begin{aligned} t_2 = t_0 + \frac{2L_x}{c} &+ \frac{\cos^2\theta}{2c}\int_0^{L_x} h_+[t_0 + x(1-\sin\theta)/c]dx \\ &+ \frac{\cos^2\theta}{2c}\int_0^{L_x} h_+[t_0 + L_x(1-\sin\theta)/c + x(1+\sin\theta)/c]dx \end{aligned} \tag{17}$$

Taking once again the derivative with respect to the start time t_0, we find

$$\begin{aligned} \frac{dt_2}{dt_0} &= 1 + \frac{\cos^2\theta}{2c}\int_0^{L_x} h'_+[t_0 + x(1-\sin\theta)/c]dx \\ &\quad + \frac{\cos^2\theta}{2c}\int_0^{L_x} h'_+[t_0 + L_x(1-\sin\theta)/c + x(1+\sin\theta)/c]dx \end{aligned} \tag{18}$$

$$\begin{aligned} &= 1 + \frac{\cos^2\theta}{2}\left\{\frac{h_+[t_0 + L_x(1-\sin\theta)/c] - h_+(t_0)}{1-\sin\theta}\right. \\ &\quad \left. + \frac{h_+[t_0 + L_x(1-\sin\theta)/c + L_x(1+\sin\theta)/c] - h_+[t_0 + L_x(1-\sin\theta)/c]}{1+\sin\theta}\right\} \end{aligned} \tag{19}$$

$$\begin{aligned} &= 1 + \frac{1}{2}\{(1+\sin\theta)h_+[t_0 + L_x(1-\sin\theta)/c] - (1+\sin\theta)h_+(t_0) \\ &\quad + (1-\sin\theta)h_+[t_0 + 2L_x/c] - (1-\sin\theta)h_+[t_0 + L_x(1-\sin\theta)/c]\} \end{aligned} \tag{20}$$

$$= 1 + \frac{1}{2}\left\{(1-\sin\theta)\, h_+\left[t_0 + 2\frac{L_x}{c}\right] - (1+\sin\theta)\, h_+(t_0) + 2\sin\theta\, h_+\left[t_0 + \frac{L_x}{c}(1-\sin\theta)\right]\right\} \tag{21}$$

the last line (21) is the *three-term formula.*

The antenna patterns

First, we expand the three-term formula for small L_x

$$\frac{dt_2}{dt_0} = 1 + \frac{1}{2}\left\{(1-\sin\theta)\; h_+\left[t_0 + 2\frac{L_x}{c}\right] - (1+\sin\theta)\; h_+(t_0) + 2\sin\theta\; h_+\left[t_0 + \frac{L_x}{c}(1-\sin\theta)\right]\right\} \tag{22}$$

$$\approx 1 + \frac{1}{2}\left\{(1-\sin\theta)\left[h_+(t_0) + \frac{2L_x}{c}\dot{h}_+(t_0)\right] - (1+\sin\theta)\; h_+(t_0) \right.$$
$$\left. +2\sin\theta\;\left[h_+[t_0] + (1-\sin\theta)\frac{L_x}{c}\dot{h}_+(t_0)\right]\right\} \tag{23}$$

$$= 1 + \frac{L_x}{c}\cos^2\theta\dot{h}_+(t_0) \tag{24}$$

and we note that the angular factor comes from the rotation of the coordinate system, and that the whole expression can be written in coordinate-free form

$$\left.\frac{dt_2}{dt_0}\right|_{\mathrm{x-arm}} = 1 + \frac{L_x}{c}\dot{h}_{ij}\hat{e}^i_x\hat{e}^j_x, \tag{25}$$

where $\hat{e}_x$ is the unit vector in the direction of the x arm, and similarly for the y arm of an interferometer with x and y arms

$$\left.\frac{dt_2}{dt_0}\right|_{\mathrm{y-arm}} = 1 + \frac{L_y}{c}\dot{h}_{ij}\hat{e}^i_y\hat{e}^j_y, \tag{26}$$

so that the global response of the interferometers is (with $L_x = L_x = L$)

$$\frac{d\delta t}{dt_0} = \left.\frac{dt_2}{dt_0}\right|_{\mathrm{x-arm}} - \left.\frac{dt_2}{dt_0}\right|_{\mathrm{y-arm}} = \frac{L}{c}\dot{h}_{ij}\left(\hat{e}^i_x\hat{e}^j_x - \hat{e}^i_y\hat{e}^j_y\right) \tag{27}$$

Finally, integrating the last equation, we find the **differential return time**

$$\delta t = \frac{L}{c}h_{ij}\left(\hat{e}^i_x\hat{e}^j_x - \hat{e}^i_y\hat{e}^j_y\right) \tag{28}$$

Expression (28) can be recast in the simpler form

$$\delta t = \frac{1}{c}h_{ij}d_{ij} \tag{29}$$

if we define the *detector tensor* **d** with components

$$d_{ij} = L\left(\hat{e}^i_x\hat{e}^j_x - \hat{e}^i_y\hat{e}^j_y\right). \tag{30}$$

This expression can be cast in the even more compact form

$$\delta t = \frac{1}{c}\mathbf{h} : \mathbf{d} \tag{31}$$

if we use the shorthand notation $\mathbf{h} : \mathbf{d} \equiv h_{ij} d_{ij}$.

Note that the definition of the detector tensor is similar to that of the polarization tensors that we defined earlier

$$\mathbf{e}_+ = \hat{e}^i_x \hat{e}^j_x - \hat{e}^i_y \hat{e}^j_y, \quad \mathbf{e}_\times = \hat{e}^i_x \hat{e}^j_y + \hat{e}^i_y \hat{e}^j_x, \tag{32}$$

and which lead to the generic expression for GW strain

$$\mathbf{h}(t) = h_+(t)\mathbf{e}_+ + h_\times(t)\mathbf{e}_\times. \tag{33}$$

Turning now to the **differential length change**, this half the length determined by the differential return time, and eq. (31) becomes

$$\delta L = \frac{1}{2}\mathbf{h} : \mathbf{d} \tag{34}$$

Figure 2 shows the geometry of the situation we are describing. In this case the polarization tensors in the TT frame are obtained from the vectors labeled $\hat{e}^R_x$, etc., shown in the left panel. However, this is a very special choice, where the $\hat{e}^R_x$ vector is parallel the x-axis of the detector frame before the rotation that brings it into the proper sky direction. In general, the system in the TT frame is also rotated by an angle ψ, as shown in the right panel of figure 2 (see also figure 3 for another view of the rotation angles).

With this additional rotation, the polarization tensors are defined by

$$\epsilon_+ = \hat{\alpha}^i \hat{\alpha}^j - \hat{\beta}^i \hat{\beta}^j, \quad \epsilon_\times = \hat{\alpha}^i \hat{\beta}^j + \hat{\beta}^i \hat{\alpha}^j, \tag{35}$$

and since the effect of the rotation is

$$\hat{\alpha} = \hat{e}^R_x \cos\psi + \hat{e}^R_y \sin\psi \tag{36}$$

$$\hat{\beta} = -\hat{e}^R_x \sin\psi + \hat{e}^R_y \cos\psi \tag{37}$$

we find

$$\epsilon^{ij}_+ = \hat{\alpha}^i \hat{\alpha}^j - \hat{\beta}^i \hat{\beta}^j \tag{38}$$

$$= \left[(\hat{e}^R_x)^i \cos\psi + (\hat{e}^R_y)^i \sin\psi\right]\left[(\hat{e}^R_x)^j \cos\psi + (\hat{e}^R_y)^j \sin\psi\right]$$
$$- \left[-(\hat{e}^R_x)^i_x \sin\psi + (\hat{e}^R_y)^i \cos\psi\right]\left[-(\hat{e}^R_x)^j \sin\psi + (\hat{e}^R_y)^j \cos\psi\right] \tag{39}$$

$$= (\hat{e}^R_x)^i(\hat{e}^R_x)^j \cos 2\psi + (\hat{e}^R_x)^i(\hat{e}^R_y)^j \sin 2\psi + (\hat{e}^R_y)^i(\hat{e}^R_x)^j \sin 2\psi - (\hat{e}^R_y)^i(\hat{e}^R_y)^j \cos 2\psi \tag{40}$$

$$= \mathbf{e}^{ij}_+ \cos 2\psi + \mathbf{e}^{ij}_\times \sin 2\psi \tag{41}$$

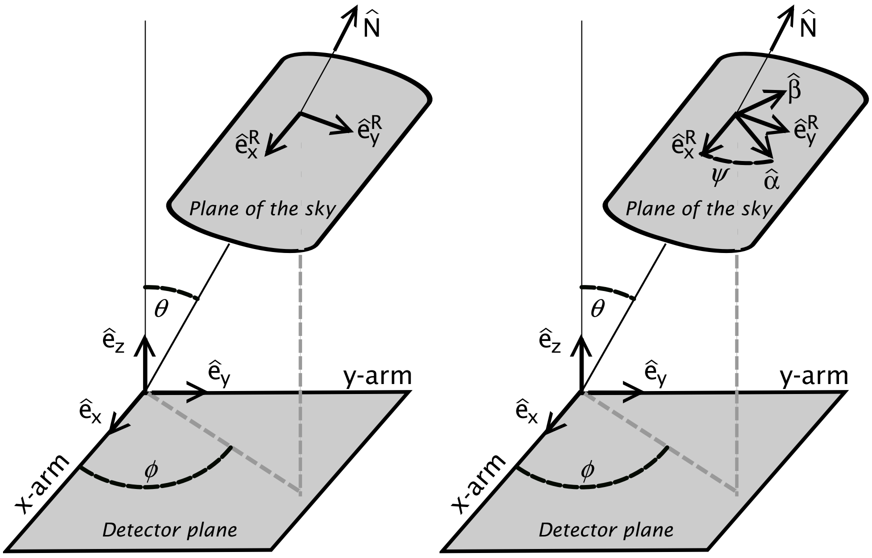

Figure 2: The relative orientation of the sky and detector frames (left panel) and the effect of a rotation by the angle ψ in the sky frame (right panel), from Sathyaprakash and Schutz [2]

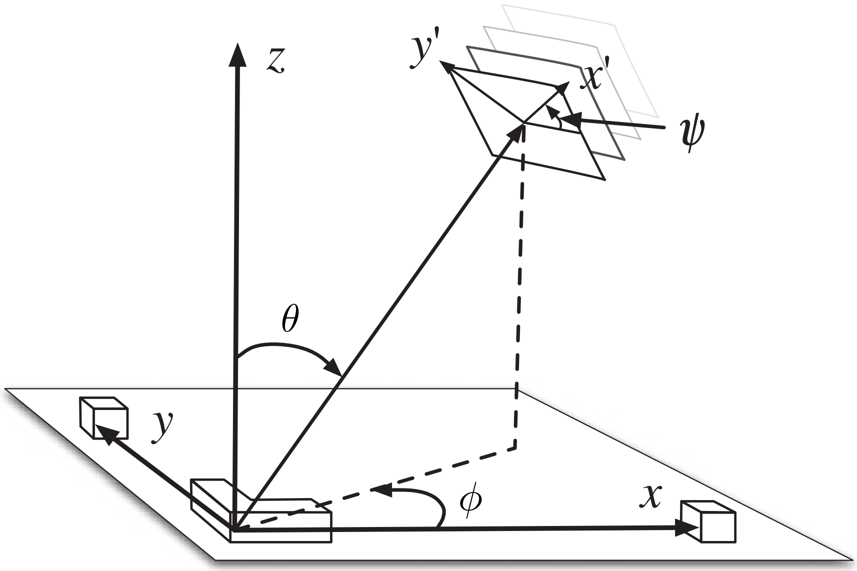

Figure 3: A representation of the individual rotations used to obtain the expression for the antenna patterns, from [1].

with a similar result for $\epsilon_\times^{ij}$:

$$\epsilon_+ = \mathbf{e}_+ \cos 2\psi + \mathbf{e}_\times \sin 2\psi \tag{42}$$

$$\epsilon_\times = -\mathbf{e}_+ \sin 2\psi + \mathbf{e}_\times \cos 2\psi \tag{43}$$

Spelling out the differential length change (34), we see that it is a function of the angles θ, ϕ, and ψ, i.e.,

$$\frac{\delta L}{L} = F_+(\theta,\phi,\psi)\; h_+(t) + F_\times(\theta,\phi,\psi)\; h_\times(t) \tag{44}$$

where the coefficients F_+ and $F_\times$ are the *antenna patterns* of the interferometer. Carrying out calculations similar to those above, it can be shown that

$$F_+ = \frac{1}{2}(1+\cos^2\theta)\cos 2\phi \cos 2\psi - \cos\theta \sin 2\phi \sin 2\psi \tag{45}$$

$$F_\times = \frac{1}{2}(1+\cos^2\theta)\cos 2\phi \sin 2\psi + \cos\theta \sin 2\phi \cos 2\psi \tag{46}$$

Figure 4 shows a graphical representation of F_+ and $F_\times$.

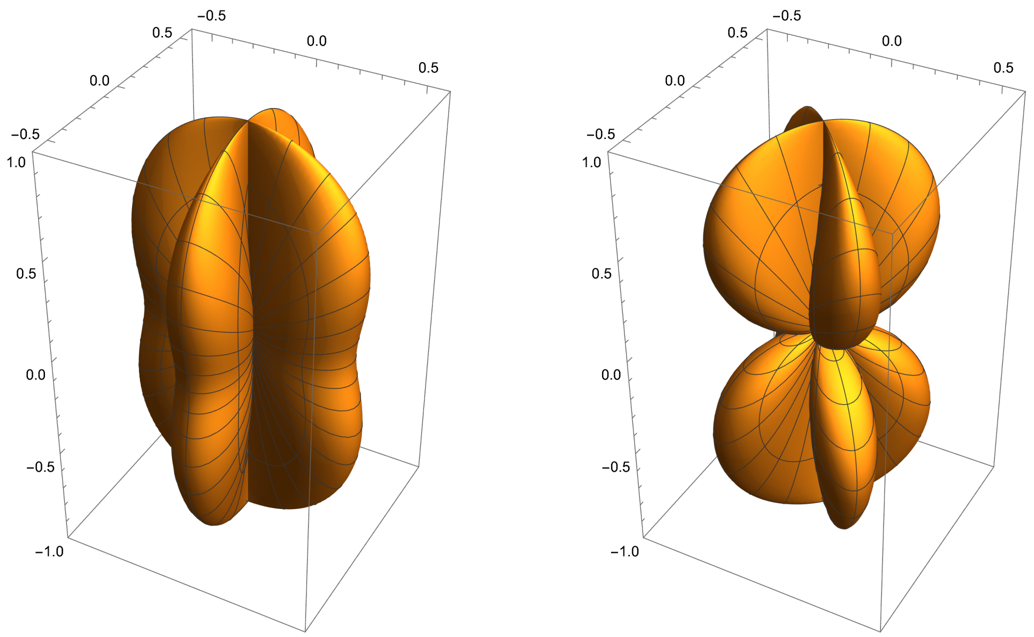

Figure 4: Antenna patterns F_+ (left) and $F_\times$ (right).

References

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