SNR calculation details

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Recall that the definition of the matched filter is

$$(s,h) = 4\ \mathrm{Re}\int_0^{+\infty} \frac{\tilde{s}(f)\tilde{h}^*(f)}{S_n^{(1)}(f)} df \tag{1}$$

and that this leads to the definition of the optimal SNR

$$(h,h) = 4\ \mathrm{Re}\int_0^{+\infty} \frac{\tilde{h}(f)\tilde{h}^*(f)}{S_n^{(1)}(f)} df = 4 \int_0^{+\infty} \frac{\tilde{h}(f)\tilde{h}^*(f)}{S_n^{(1)}(f)} df = \int_{-\infty}^{+\infty} \frac{\tilde{h}(f)\tilde{h}^*(f)}{S_n(f)} df \tag{2}$$

If $\tilde{w}(f) = \tilde{h}(f)/\sqrt{S_n(f)}$ is the whitened waveform, then

$$(h,h) = \int_{-\infty}^{+\infty} \tilde{w}(f)\tilde{w}^*(f) df = \int_{-\infty}^{+\infty} w(t)w^*(t) dt \to \sum_n w_n w_n^* \Delta t \tag{3}$$

where the sum ranges over the whole region where $w(t) \to w_n = w(n\Delta t)$ is not negligible.

The calculation in the time domain requires the evaluation of the inverse Fourier transform starting from the DFT. Recall that

$$\tilde{s}(f) = \int_{-\infty}^{+\infty} s(t) e^{-2ift} dt \to \sum_n s_n e^{-2ik\Delta f\ n\Delta t} \Delta t \tag{4}$$

where $T = N\Delta t$, $\Delta f = 1/T$, $\Delta t \Delta f = 1/N$, and therefore

$$\tilde{s}(f) \approx \sum_n s_n e^{-2ikn/N} \Delta t = \mathrm{DFT}[s]\Delta t \tag{5}$$

where the DFT (and its inverse) is defined as follows

$$\mathrm{DFT}[s] = \sum_{n=0}^{N-1} s_n e^{-2ikn/N}, \quad \mathrm{IDFT}[\tilde{s}] = \frac{1}{N}\sum_{n=0}^{N-1} \tilde{s}_k e^{2ikn/N}. \tag{6}$$

Similarly, we find

$$s(t) = \int_{-\infty}^{+\infty} \tilde{s}(f) e^{2ift} df \to \sum_k \tilde{s}_k e^{2ik\Delta f\ n\Delta t} \Delta f = N\ \mathrm{IDFT}[\tilde{s}]\Delta f \tag{7}$$

Thus,

$$s(t) = \int_{-\infty}^{+\infty} df\ e^{2ift} \int_{-\infty}^{+\infty} dt\ e^{-2ift} s(t) = \mathrm{IDFT}[\mathrm{DFT}[s]] N\Delta f \Delta t = \mathrm{IDFT}[\mathrm{DFT}[s]] \tag{8}$$