Antenna patterns - 2

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Here I start from the formula for the observed strain that was derived in the handout *Antenna patterns*

$$\frac{\delta L}{L} = F_+(\theta,\phi,\psi)\ h_+(t) + F_\times(\theta,\phi,\psi)\ h_\times(t) \tag{1}$$

where

$$F_+ = \frac{1}{2}(1+\cos^2\theta)\cos 2\phi \cos 2\psi - \cos\theta \sin 2\phi \sin 2\psi \tag{2}$$

$$F_\times = \frac{1}{2}(1+\cos^2\theta)\cos 2\phi \sin 2\psi + \cos\theta \sin 2\phi \cos 2\psi \tag{3}$$

The Fourier transform of the signal is

$$\frac{\delta \tilde{L}}{L} = F_+(\theta,\phi,\psi)\ \tilde{h}_+(f) + F_\times(\theta,\phi,\psi)\ \tilde{h}_\times(f) \tag{4}$$

therefore the optimal power SNR is

$$\rho^2 = 4\int_0^{+\infty} \frac{1}{S_n^{(1)}(f)} |F_+(\theta,\phi,\psi)\ \tilde{h}_+(f) + F_\times(\theta,\phi,\psi)\ \tilde{h}_\times(f)|^2 df \tag{5}$$

$$= 4\int_0^{+\infty} \frac{1}{S_n^{(1)}(f)} \Big[F_+^2(\theta,\phi,\psi)\ |\tilde{h}_+(f)|^2 \tag{6}$$

$$+F_\times^2(\theta,\phi,\psi)\ |\tilde{h}_\times(f)|^2 + 2F_+(\theta,\phi,\psi)F_\times(\theta,\phi,\psi)\ \mathrm{Re}\ \tilde{h}_+(f)\tilde{h}_\times^*(f) \Big]\ df \tag{7}$$

Now notice that

$$F_+^2 = \frac{1}{4}(1+\cos^2\theta)^2\cos^2 2\phi\cos^2 2\psi + \cos^2\theta\sin^2 2\phi\sin^2 2\psi - \frac{1}{4}(1+\cos^2\theta)\cos\theta\sin 4\phi\sin 4\psi \quad (8)$$

$$F_\times^2 = \frac{1}{4}(1+\cos^2\theta)^2\cos^2 2\phi\sin^2 2\psi + \cos^2\theta\sin^2 2\phi\cos^2 2\psi + \frac{1}{4}(1+\cos^2\theta)\cos\theta\sin 4\phi\sin 4\psi \quad (9)$$

$$\begin{aligned} F_+F_\times &= \frac{1}{8}(1+\cos^2\theta)^2\cos^2 2\phi\sin 4\psi + \frac{1}{8}(1+\cos^2\theta)\cos\theta\sin 4\phi\cos^2 2\psi \\ &- \frac{1}{8}(1+\cos^2\theta)\cos\theta\sin 4\phi\sin^2 2\psi - \frac{1}{2}\cos^2\theta\sin^2 2\phi\sin 4\psi \\ &= \frac{1}{8}(1+\cos^2\theta)^2\cos 4\phi\sin 4\psi + \frac{1}{8}(1+\cos^2\theta)\cos\theta\sin 4\phi\cos 4\psi - \frac{1}{2}\cos^2\theta\sin^2 2\phi\sin 4\psi \end{aligned} \quad (10)$$

This means that when we average over ψ, i.e., we assume that wave polarizations are random, we find

$$\langle F_+^2\rangle = \frac{1}{8}(1+\cos^2\theta)^2\cos^2 2\phi + \frac{1}{2}\cos^2\theta\sin^2 2\phi \quad (11)$$

$$\langle F_\times^2\rangle = \frac{1}{8}(1+\cos^2\theta)^2\cos^2 2\phi + \frac{1}{2}\cos^2\theta\sin^2 2\phi \quad (12)$$

$$\langle F_+F_\times\rangle = 0$$

so that after averaging

$$\langle\rho^2\rangle = 2P(\theta,\phi)\int_0^{+\infty}\frac{|\tilde{h}(f)|^2}{S_n^{(1)}(f)}df \quad (13)$$

where

$$|\tilde{h}(f)|^2 = |\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2. \quad (14)$$

and where

$$P(\theta,\phi) = \frac{1}{4}(1+\cos^2\theta)^2\cos^2 2\phi + \cos^2\theta\sin^2 2\phi \quad (15)$$

is the antenna power pattern (see figure 1). This also explains why it is common to use the combination (14) to define the *root sum of squares*

$$h_{\rm rss} = \sqrt{\int_0^{+\infty}\left(|\tilde{h}_+(f)|^2 + |\tilde{h}_\times(f)|^2\right)df} \quad (16)$$

which – assuming whitened strains – corresponds to the strain in eq. (13).

Now, we can connect these results to the detection volume. In the time domain, the amplitude of the gravitational wave decays as the inverse of the distance to the source, i.e.,

$$h = \frac{r_s}{r}h_s \quad (17)$$

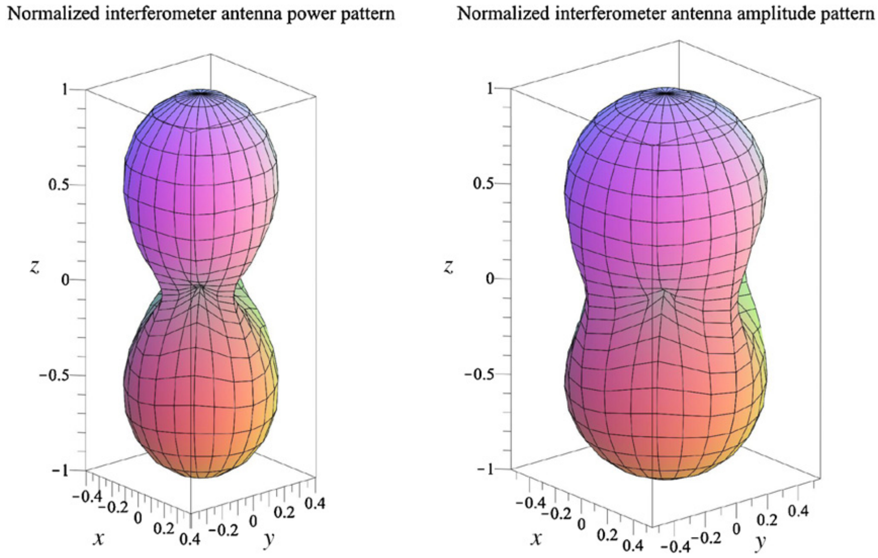

Figure 1: The antenna power pattern (left panel) and its square root (amplitude pattern: right panel) of a single interferometer oriented with axes in the x–y plane, averaged over polarizations of the incoming wave. The amplitude pattern represents the shape of the detection volume of the instrument, or its maximum detection reach in different directions (from [1]

where h_s is the amplitude at the standard distance r_s. The same holds for the Fourier transform

$$\tilde{h}(f) = \frac{r_s}{r}\tilde{h}_s(f) \tag{18}$$

and we can use this to explicitly indicate the dependence on the source distance in eq. (13)

$$\langle\rho^2\rangle = \frac{2}{r^2}P(\theta,\phi)\int_0^{+\infty}\frac{|r_s\tilde{h}_s(f)|^2}{S_n(f)}df \tag{19}$$

When we set the standard threshold for visibility of a source at $\rho_{\rm min} = \sqrt{\langle\rho^2\rangle} = 8$, we find the corresponding distance, and in particular for the maximum of $P(\theta,\phi)$ which is obtained for $\theta = 0$ or $\theta = \pi$, $P(0,\phi) = 1$, we obtain the visibility distance

$$D_V = \frac{1}{\rho_{\rm min}}\sqrt{2\int_0^{+\infty}\frac{|r_s\tilde{h}_r(f)|^2}{S_n(f)}df} \tag{20}$$

Notice that all the complex details of filtering and the detector noise curve are hidden in the single parameter D_V.

When several detectors are operating in a common network, the total SNR is just the sum of the individual SNRs

$$\rho_N^2 = \sum_{k=1,N_D} \rho_k^2 \tag{21}$$

where N_D is the number of detectors in operation. Therefore, after averaging over the polarization angle, we find

$$\langle \rho_N^2 \rangle = \frac{2}{r^2} \sum_{k=1,N_D} P_k(\theta,\phi) \int_0^{+\infty} \frac{|r_s \tilde{h}_s(f)|^2}{S_{n,k}(f)} df \tag{22}$$

This distorts the antenna power patterns and makes the antenna response more isotropic.

References

[1] Bernard F Schutz. Networks of gravitational wave detectors and three figures of merit. *Classical and Quantum Gravity*, 28(12):125023, 2011.