Notes on the Li & Paczyński "kilonova paper"

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December 3, 2024

In this short handout I provide additional notes and quick derivations of some equations in the Li & Paczyński paper [1] (LP).

- LP denote the speed with V, but we also need a symbol for volume, so here I use v for speed and V for volume.
- Here I use the standard convention that U is the energy and u is the energy density.
- LP denote the entropy per unit mass with the symbol S; following the standard conventions I use S for entropy and S/M for the entropy per unit mass.
- equation (1) is correct only for long times, as it stands it diverges for $t \to 0$. The point is that the density decreases with increasing radius starting from $R(t = 0) = R_0$, and $R(t) = R_0 + vt$. This means that

$$\rho(t) = \frac{3M}{4\pi v^3}(t + R_0/v)^{-3} \tag{1}$$

  for $t \geq 0$.

- From the equations for the energy density of blackbody radiation it can be shown that the energy density is

$$u = 3P = aT^4 \tag{2}$$

  where P is the pressure, T is the temperature of the photon gas, and

$$a = \frac{8\pi^5 k_B^4}{15h^3c^3} \approx 7.56 \times 10^{-16}\ \mathrm{J\ K^{-4}m^{-3}}. \tag{3}$$

  The parameter a is related to the Stefan-Boltzmann constant: $\sigma = ac/4$.

  It can also be shown that the entropy is

$$S = \frac{4U}{3T} = \frac{4}{3}aVT^3 \tag{4}$$

Then, the internal energy change per unit mass in the LP paper (equation 3 in the paper) is

$$\frac{TdS}{M} = \frac{dU}{M} = d\left(\frac{U/V}{M/V}\right) = d\left(\frac{u}{\rho}\right) = \frac{du}{\rho} + u\, d\left(\frac{1}{\rho}\right). \tag{5}$$

Using equation 1 in the paper (valid for long times $t \gg R_0/v$)

$$\rho = \frac{3M}{4\pi v^3} t^{-3}, \tag{6}$$

we find

$$\frac{dU}{M} = \frac{4\pi v^3}{3M}\left(t^3 du + 3ut^2 dt\right). \tag{7}$$

- Equation 5 for radioactive heating in LP, is obtained as follows, using the rate $\lambda_{\rm rad} = t_{\rm rad}^{-1}$ instead of $t_{\rm rad}$. A uniform distribution of $\ln t_{\rm rad}$ corresponds to a power law distribution of t which which corresponds to the probability

$$\sim t^{-1} dt = \lambda \left| d\left(\frac{1}{\lambda}\right)\right| = \frac{d\lambda}{\lambda} = d\ln\lambda, \tag{8}$$

i.e., we obtain a uniform distribution in $\ln\lambda$. Then we have to compute the integral

$$\epsilon = fc^2 \int_{\lambda_{\rm min}}^{\lambda_{\rm max}} \lambda e^{-\lambda t}\frac{d\lambda}{\lambda} = -\left.\frac{fc^2}{t} e^{-\lambda t}\right|_{\lambda_{\rm min}}^{\lambda_{\rm max}} \approx \frac{fc^2}{t}. \tag{9}$$

This is the heat generation rate *per unit mass per unit time*.

- Radiative losses. Substituting the expressions for u, $R = vt$, $\rho = \frac{3M}{4\pi v^3}t^{-3}$ and $\sigma = ac/4$ in the expression $L = 4\pi R^2 F$ (radiated power) we find the heat loss per unit time

$$L = 4\pi R^2 F = 4\pi R^2 \frac{\sigma T^4}{\kappa\rho R} = 4\pi R^2 \frac{acT^4}{4\kappa R}\frac{4\pi v^3 t^3}{3M} = \frac{4\pi^2 c}{3\kappa M}(aT^4)(v^4 t^4) = \frac{4\pi^2 c v^4}{3\kappa M} u t^4, \tag{10}$$

which is equation 6 in LP.

- If $\kappa\rho R = 1$, then we obtain equation 7 in LP as follows:

$$R = vt_c = \frac{1}{\kappa\rho} = \frac{4\pi v^3 t_c^3}{3\kappa M}, \tag{11}$$

i.e.,

$$t_c = \left(\frac{3\kappa M}{4\pi v^2}\right)^{1/2}. \tag{12}$$

- Starting from the heat balance

$$dU = M\epsilon dt - dQ \quad \Rightarrow \quad \frac{1}{M}\frac{dU}{dt} = \frac{L}{M} = \epsilon - T\frac{d(S/M)}{dt} \tag{13}$$

we find

$$\frac{1}{M}\left(\frac{4\pi^2 c v^4}{3\kappa M} u t^4\right) = \epsilon - \left(\frac{4\pi v^3}{3M}\right)\left(t^3 \frac{du}{dt} + 3ut^2\right), \tag{14}$$

and finally we obtain eq. (9) in LP

$$t^3 \frac{du}{dt} + 3ut^2 = \left(\frac{3M}{4\pi v^3}\right)\epsilon - \left(\frac{\pi c v}{\kappa M}\right) u t^4. \tag{15}$$

References

[1] Li-Xin Li and Bohdan Paczyński. Transient events from neutron star mergers. *The Astrophysical Journal*, 507(1):L59, 1998.