The Schwarzschild geometry

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Compact objects, whose binary systems are the most frequent sources of gravitational waves, have strong gravitational fields that, in the special case of non-rotating objects, can be described by the Schwarzschild metric. Although in this course we do not deal with compact objects in detail, we introduce here this important metric, following the general treatment in [1].

1 The Schwarzschild metric

As a first step, we determine the most general stationary isotropic metric around a massive spherical object. **This metric must depend only on the three rotational invariants of the spacelike coordinate**, i.e., on

$$r^2 = \mathbf{x} \cdot \mathbf{x}, \quad d\mathbf{x} \cdot d\mathbf{x}, \quad \mathbf{x} \cdot d\mathbf{x}, \tag{1}$$

i.e., the line element must have the form

$$ds^2 = A(t,r)dt^2 - B(t,r)dt(\mathbf{x} \cdot d\mathbf{x}) - C(t,r)(\mathbf{x} \cdot d\mathbf{x})^2 - D(t,r)(d\mathbf{x} \cdot d\mathbf{x}) \tag{2}$$

where A, B, C, and D are **arbitrary functions** of t and r.

When we write the spatial coordinates in polar form

$$x^1 = r \sin\theta \cos\phi, \quad x^2 = r \sin\theta \sin\phi, \quad x^3 = r \cos\theta, \tag{3}$$

we find

$$dx^1 = \sin\theta \cos\phi \, dr + r \cos\theta \cos\phi \, d\theta - r \sin\theta \sin\phi \, d\phi \tag{4a}$$
$$dx^2 = \sin\theta \sin\phi \, dr + r \cos\theta \sin\phi \, d\theta + r \sin\theta \cos\phi \, d\phi \tag{4b}$$
$$dx^3 = \cos\theta \, dr - r \sin\theta \, d\theta \tag{4c}$$

and therefore

$$\begin{aligned} \mathbf{x} \cdot d\mathbf{x} = & \sin^2\theta \cos^2\phi \, r dr + \cos\theta \sin\theta \cos^2\phi \, r d\theta - \sin^2\theta \cos\phi \sin\phi \, r d\phi \\ & + \sin^2\theta \sin^2\phi \, r dr + \cos\theta \sin\theta \sin^2\phi \, r d\theta + \sin\theta^2 \cos\phi \sin\phi \, r d\phi \\ & + \cos\theta^2 \, r dr - r \sin\theta \cos\theta \, d\theta = r dr \end{aligned} \tag{5a}$$

$$d\mathbf{x} \cdot d\mathbf{x} = dr^2 + r^2 \, d\theta^2 + r^2 \sin^2\theta \, d\phi^2. \tag{5b}$$

Therefore,

$$ds^2 = A(t,r)dt^2 - B(t,r)r\,dt\,dr - C(t,r)r^2dr^2 - D(t,r)(dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2). \quad (6)$$

Now, we carry out a sequence of nontrivial steps:

- we collect terms

$$ds^2 = A(t,r)dt^2 - B(t,r)r\,dt\,dr - \left[C(t,r)r^2 + D(t,r)\right]dr^2 - D(t,r)(r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2), \quad (7)$$

- we redefine the metric by absorbing the terms in r into the definitions of the coefficients

$$ds^2 = A(t,r)dt^2 - B(t,r)dt\,dr - C(t,r)dr^2 - D(t,r)(d\theta^2 + \sin^2\theta\,d\phi^2), \quad (8)$$

- we define a new radial coordinate $\bar{r}^2 = D(t,r)$, so that

$$ds^2 = A(t,\bar{r})dt^2 - B(t,\bar{r})dt\,d\bar{r} - C(t,\bar{r})d\bar{r}^2 - \bar{r}^2(d\theta^2 + \sin^2\theta\,d\phi^2), \quad (9)$$

  (with a corresponding redefinition of the coefficients),

- we diagonalize the metric by defining a new timelike coordinate $\bar{t}$, such that

$$d\bar{t} = \Phi(t,\bar{r})\left[A(t,\bar{r})dt - \frac{1}{2}B(t,\bar{r})d\bar{r}\right], \quad (10)$$

  and therefore

$$d\bar{t}^2 = \Phi^2\left[A^2dt^2 - AB\,dt\,d\bar{r} + \frac{1}{4}B^2d\bar{r}^2\right], \quad (11)$$

  from which we find

$$Adt^2 - B\,dt\,d\bar{r} = \frac{1}{A\Phi^2}d\bar{t}^2 - \frac{1}{4A}B^2d\bar{r}^2. \quad (12)$$

- we redefine the coefficients so that

$$\bar{A} = \frac{1}{A\Phi^2}, \quad \bar{B} = C + \frac{B^2}{4A}, \quad (13)$$

  and find

$$ds^2 = \bar{A}d\bar{t}^2 - \bar{B}d\bar{r}^2 - \bar{r}^2(d\theta^2 + \sin^2\theta\,d\phi^2), \quad (14)$$

- we remove bars, to obtain the general expression for the isotropic, spherically symmetric metric

$$ds^2 = A(t,r)dt^2 - B(t,r)dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2), \quad (15)$$

- finally, we assume that the metric does not depend on t, i.e., that the metric *stationary*

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{16}$$

The following comments are in order:

- the metric is stationary because there is no dependence on t, but this does not mean that there are no dynamic processes; in case there are no ongoing dynamic processes, we require the metric to be independent of the direction of time, i.e. invariant with respect to the transformation $t \to -t$, and in this case the metric is called *static*;
- an example of a stationary metric is that produced by a rotating star, where a reversal of time direction changes the direction of the angular velocity but the metric components remain unchanged;
- the line element (16) is already invariant with respect to the transformation $t \to -t$, and therefore it is not only stationary but also static;
- for fixed t, r the line element (16) describes the geometry of 2-spheres.

2 Solution of empty-space field equations

The metric tensor associated with the line element (16) is diagonal and has just 4 non-vanishing components

$$g_{00} = A(r), \quad g_{11} = -B(r), \quad g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta. \tag{17}$$

Correspondingly, the contravariant metric tensor has components

$$g^{00} = \frac{1}{A(r)}, \quad g^{11} = -\frac{1}{B(r)}, \quad g^{22} = -\frac{1}{r^2}, \quad g^{33} = -\frac{1}{r^2\sin^2\theta}. \tag{18}$$

We can use these expressions to compute the Ricci tensor and find the solution of Einstein's equations in vacuum ($T^{\mu\nu} = 0$)

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right) = 0 \tag{19}$$

Using equations (18) and with the help of the *Diagonal metric worksheet*, we can find out all the nonvanishing components of the Ricci tensor, i.e.,

$$R_{00} = -\frac{A''}{2B} + \frac{A'}{4B}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{A'}{rB} \tag{20a}$$

$$R_{11} = \frac{A''}{2A} - \frac{A'}{4A}\left(\frac{A'}{A} + \frac{B'}{B}\right) - \frac{B'}{rB} \tag{20b}$$

$$R_{22} = \frac{1}{B} - 1 + \frac{r}{2B}\left(\frac{A'}{A} - \frac{B'}{B}\right) \tag{20c}$$

$$R_{33} = R_{22}\sin^2\theta \tag{20d}$$

From eqs. (20a) and (20b), we find

$$A'B + AB' = (AB)' = 0 \tag{21}$$

and therefore

$$AB = \text{constant} \equiv \alpha \tag{22}$$

Substituting $B = \alpha/A$ in (20c), we find

$$A + rA' = \alpha \tag{23}$$

which we solve to obtain

$$rA = \alpha(r + k) \tag{24}$$

with k another integration constant. Finally,

$$A(r) = \alpha\left(1 + \frac{k}{r}\right), \quad B(r) = \left(1 + \frac{k}{r}\right)^{-1} \tag{25}$$

In the weak field limit we must retrieve the Newtonian result, i.e.,

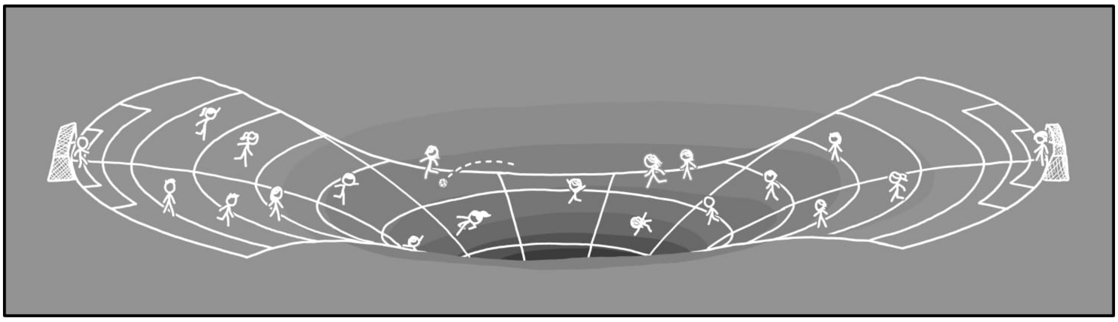
$$\frac{A(r)}{c^2} \to \left(1 + \frac{2\Phi}{c^2}\right) \tag{26}$$

where Φ is the Newtonian gravitational potential $\Phi = -GM/r$, and therefore $\alpha = 1$, $k = -2GM/c^2$, and the Schwarzschild metric is given by the line element

$$ds^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2\sin^2\theta d\phi^2. \tag{27}$$

References

[1] Michael Paul Hobson, George P Efstathiou, and Anthony N Lasenby. *General relativity: an introduction for physicists.* Cambridge University Press, 2006.



SPACETIME SOCCER GOT A LOT OF CRITICISM FOR HOW MANY PLAYERS FELL INTO THE GRAVITY WELL, BUT WHAT ULTIMATELY DOOMED IT WAS THE ADVANCED MATHEMATICS REQUIRED TO FIGURE OUT THE OFFSIDES RULE.