Working with connection coefficients

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In this handout we consider a few simple examples on connection coefficients.

- **A two–dimensional example**

  Here, we calculate the connection coefficients for two–dimensional Euclidean space in polar coordinates. There are 8 components, but overall, taking the symmetry of the covariant indices into account, there are 6 independent Γ values.

  The coordinate transformation is

$$x = r\cos\theta \tag{1a}$$

$$y = r\sin\theta \tag{1b}$$

  and recalling the formula for the covariant basis vectors

$$\boldsymbol{\varepsilon}_i = \frac{\partial x}{\partial x'^i}\hat{\mathbf{e}}_x + \frac{\partial y}{\partial x'^i}\hat{\mathbf{e}}_y \tag{2}$$

  we find

$$\boldsymbol{\varepsilon}_r = \frac{\partial x}{\partial r}\hat{\mathbf{e}}_x + \frac{\partial y}{\partial r}\hat{\mathbf{e}}_y = \cos\theta\hat{\mathbf{e}}_x + \sin\theta\hat{\mathbf{e}}_y \tag{3a}$$

$$\boldsymbol{\varepsilon}_\theta = \frac{\partial x}{\partial \theta}\hat{\mathbf{e}}_x + \frac{\partial y}{\partial \theta}\hat{\mathbf{e}}_y = -r\sin\theta\hat{\mathbf{e}}_x + r\cos\theta\hat{\mathbf{e}}_y \tag{3b}$$

  and

$$\hat{\mathbf{e}}_x = \cos\theta\boldsymbol{\varepsilon}_r - \frac{1}{r}\sin\theta\boldsymbol{\varepsilon}_\theta \tag{4a}$$

$$\hat{\mathbf{e}}_y = \sin\theta\boldsymbol{\varepsilon}_r + \frac{1}{r}\cos\theta\boldsymbol{\varepsilon}_\theta \tag{4b}$$

  Next, recalling the definition

$$\frac{\partial \boldsymbol{\varepsilon}_k}{\partial x^j} = \Gamma^\mu_{jk}\boldsymbol{\varepsilon}_\mu \tag{5}$$

and noticing that the Cartesian basis vectors are independent from r and θ, we compute the corresponding derivatives:

$$\frac{\partial \boldsymbol{\varepsilon}_r}{\partial r} = 0 \tag{6a}$$
$$\frac{\partial \boldsymbol{\varepsilon}_r}{\partial \theta} = -\sin\theta \hat{\mathbf{e}}_x + \cos\theta \hat{\mathbf{e}}_y = \frac{1}{r}\boldsymbol{\varepsilon}_\theta \tag{6b}$$
$$\frac{\partial \boldsymbol{\varepsilon}_\theta}{\partial r} = -\sin\theta \hat{\mathbf{e}}_x + \cos\theta \hat{\mathbf{e}}_y = \frac{1}{r}\boldsymbol{\varepsilon}_\theta \tag{6c}$$
$$\frac{\partial \boldsymbol{\varepsilon}_\theta}{\partial \theta} = -r\cos\theta \hat{\mathbf{e}}_x - r\sin\theta \hat{\mathbf{e}}_y = -r\boldsymbol{\varepsilon}_r \tag{6d}$$

and finally, we find

$$\Gamma^r_{rr} = \Gamma^\theta_{rr} = 0 \tag{7a}$$
$$\Gamma^r_{r\theta} = \Gamma^r_{\theta r} = 0 \tag{7b}$$
$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \tag{7c}$$
$$\Gamma^r_{\theta\theta} = -r; \quad \Gamma^\theta_{\theta\theta} = 0 \tag{7d}$$

- **Utilizing the formula based on the metric tensor**

  Recall that

$$\Gamma^n_{ij} = \frac{1}{2} g^{nk}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right) \tag{8a}$$
$$= \frac{1}{2} g^{nk}\left(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}\right) \tag{8b}$$

  and notice that in polar coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{9}$$

  so that

$$[g_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{10}$$

  and

$$[g^{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix} \tag{11}$$

  Then,

$$\Gamma^\theta_{r\theta} = \frac{1}{2} g^{\theta k}\left(\frac{\partial g_{rk}}{\partial \theta} + \frac{\partial g_{\theta k}}{\partial r} - \frac{\partial g_{r\theta}}{\partial x^k}\right) = \frac{1}{2} g^{\theta k}\left(0 + \frac{\partial g_{\theta\theta}}{\partial r}\delta^\theta_k - 0\right) = \frac{1}{2} g^{\theta\theta}\frac{\partial g_{\theta\theta}}{\partial r} = \frac{1}{r} \tag{12}$$

  and similar formulas for the other Γs.