

Geodesics in curved spaces

Edoardo Milotti

October 5, 2025

Here, we consider geodesics on the surface of a sphere embedded in 3D Euclidean space.

1 The case of a 2D sphere in 3D Euclidean space

1.1 The metric tensor

Here we repeat a simple calculation that we already met in the handout “Special Relativity and notation”. We start from the equations for Cartesian coordinates in terms of spherical coordinates as follows:

$$x = R \sin \theta \cos \varphi \quad (1a)$$

$$y = R \sin \theta \sin \varphi \quad (1b)$$

$$z = R \cos \theta \quad (1c)$$

from which we find

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \varphi} d\varphi = \sin \theta \cos \varphi dr + r \cos \theta \cos \varphi d\theta - r \sin \theta \sin \varphi d\varphi \quad (2a)$$

$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \varphi} d\varphi = \sin \theta \sin \varphi dr + r \cos \theta \sin \varphi d\theta + r \sin \theta \cos \varphi d\varphi \quad (2b)$$

$$dz = \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \theta} d\theta + \frac{\partial z}{\partial \varphi} d\varphi = \cos \theta dr - r \sin \theta d\theta \quad (2c)$$

Finally, squaring and summing, one finds the space interval:

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2 \theta d\varphi^2 = g_{ij} dx^i dx^j \quad (3)$$

and therefore

$$[g_{ij}] = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2 \theta \end{pmatrix} \quad (4)$$

and

$$[g^{ij}] = \begin{pmatrix} 1/R^2 & 0 \\ 0 & 1/R^2 \sin^2 \theta \end{pmatrix} \quad (5)$$

1.2 The connection coefficients

This is a 2D space, therefore the connection coefficient has 8 values. Given the symmetry of the covariant indexes, the total number of independent components is 6. Using the formula

$$\Gamma_{ij}^n = \frac{1}{2} g^{nk} (\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}) \quad (6)$$

and the fact that the only nonzero derivative of the metric tensor components is

$$\partial_\theta g_{\varphi\varphi} = 2R^2 \sin \theta \cos \theta \quad (7)$$

we find

$$\Gamma_{\theta\theta}^\theta = \Gamma_{\theta\varphi}^\theta = \Gamma_{\varphi\theta}^\theta = \Gamma_{\theta\theta}^\varphi = \Gamma_{\varphi\varphi}^\varphi = 0 \quad (8)$$

$$\Gamma_{\theta\varphi}^\varphi = \Gamma_{\varphi\theta}^\varphi = \frac{\cos \theta}{\sin \theta} \quad (9)$$

$$\Gamma_{\varphi\varphi}^\theta = -\sin \theta \cos \theta \quad (10)$$

1.3 Solution of the geodesic equations

Starting from the generic geodesic equation in space

$$\frac{d^2 x^\mu}{ds^2} + \frac{dx^\sigma}{ds} \frac{dx^\nu}{ds} \Gamma_{\sigma\nu}^\mu = 0, \quad (11)$$

we find

$$\frac{d^2 \theta}{ds^2} + \frac{dx^\sigma}{ds} \frac{dx^\nu}{ds} \Gamma_{\sigma\nu}^\theta = 0 \quad (12a)$$

$$\frac{d^2 \varphi}{ds^2} + \frac{dx^\sigma}{ds} \frac{dx^\nu}{ds} \Gamma_{\sigma\nu}^\varphi = 0 \quad (12b)$$

and given the results of the previous section, the summations can be explicitly carried out and we find

$$\frac{d^2 \theta}{ds^2} - \left(\frac{d\varphi}{ds} \right)^2 \sin \theta \cos \theta = 0 \quad (13a)$$

$$\frac{d^2 \varphi}{ds^2} + 2 \frac{d\theta}{ds} \frac{d\varphi}{ds} \frac{\cos \theta}{\sin \theta} = 0 \quad (13b)$$

It is easy to see that we can rearrange equation (13b)

$$\frac{d^2 \theta}{ds^2} - \left(\frac{d\varphi}{ds} \right)^2 \sin \theta \cos \theta = 0 \quad (14a)$$

$$\frac{d}{ds} \left(\sin^2 \theta \frac{d\varphi}{ds} \right) = 0 \quad (14b)$$

which is a system of two second-order differential equation and requires 4 integration constants. Integrating eq. (14b) we obtain

$$\sin^2 \theta \frac{d\varphi}{ds} = \frac{C}{R} \quad (15)$$

where the integration constant C/R is chosen so that C is a nondimensional constant and R sets the length scale. Then, equation (14a) becomes

$$\frac{d^2 \theta}{ds^2} = \frac{1}{2} \frac{d}{d\theta} \left(\frac{d\theta}{ds} \right)^2 = \frac{C^2 \cos \theta}{R^2 \sin^3 \theta} = -\frac{1}{2} \frac{C^2}{R^2} \frac{d}{d\theta} \left(\frac{1}{\sin^2 \theta} \right) \quad (16)$$

which can be integrated directly to give

$$\left(\frac{d\theta}{ds} \right)^2 = A - \frac{C^2}{R^2} \left(\frac{1}{\sin^2 \theta} \right) \quad (17)$$

where A is another integration constant. We obtain a real solution by taking a starting point for the geodesic that is on the equator of the sphere, i.e., $\theta(s = 0) = \pi/2$, and setting $A = C^2/R^2$, so that

$$\left(\frac{d\theta}{ds}\right)^2 = 0 \quad (18)$$

i.e., the geodesic coincides with the equator, while eq. (15) becomes

$$\frac{d\varphi}{ds} = \frac{C}{R} \quad (19)$$

We still have 1 degree of freedom to use (1 integration constant) and we note that we can simplify the problem by choosing a starting point with $\varphi(0) = 0$ so that the final solution is

$$\theta(s) = \frac{\pi}{2}; \quad \varphi(s) = \frac{C}{R}s \quad (20)$$

i.e., the solution is just the greatest circle coinciding with the equator. Finally, requiring periodicity, we set $C = \pm 1$ (the final degree of freedom) and

$$\varphi(s) = \pm \frac{s}{R} \quad (21)$$

such that the periodic boundary condition $\varphi(0) = \pm\varphi(2\pi R)$ is satisfied.

With different choices of starting points we obtain different geodesics, but given that we can always rotate the sphere to match the conditions of this solution, we see that the set of great circles coincides with all the solutions of the geodesic equations on the surface of the sphere.