Geodesics in curved spacetime

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Geodesics can be defined either as extremal curves with the shortest possible path length or as the curves that depart as little as possible from straightness (they are "locally straight").

1 The geodesic equation – 1

Here, we obtain the geodesic equation looking at geodesics as the trajectories that depart as little as possible from straightness. A point mass moving along a geodesic has its velocity vector $U = dX/d\tau$ always tangent to the path. A geodesic is "locally straight", therefore we require that the velocity vector does not change locally, i.e., that the **absolute derivative** vanishes

$$0 = \frac{dU}{d\tau} = \frac{d}{d\tau}(U^\mu \boldsymbol{\varepsilon}_\mu) = \frac{dU^\mu}{d\tau}\boldsymbol{\varepsilon}_\mu + U^\mu \frac{d\boldsymbol{\varepsilon}_\mu}{d\tau}. \tag{1}$$

Since

$$\frac{d\boldsymbol{\varepsilon}_\mu}{d\tau} = \frac{\partial \boldsymbol{\varepsilon}_\mu}{\partial x^\nu}\frac{dx^\nu}{d\tau} = U^\nu \Gamma^\sigma_{\mu\nu}\boldsymbol{\varepsilon}_\sigma \tag{2}$$

we find

$$\frac{d^2x^\mu}{d\tau}\boldsymbol{\varepsilon}_\mu + \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\Gamma^\sigma_{\mu\nu}\boldsymbol{\varepsilon}_\sigma = 0 \tag{3}$$

and finally, exchanging σ and μ

$$\frac{d^2x^\mu}{d\tau} + \frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau}\Gamma^\mu_{\sigma\nu} = 0 \tag{4}$$

which is the **geodesic equation**.

The second term in the geodesic equation contains the basic information about curvature: we see as the connection coefficients approach zero, and therefore we are close to a flat spacetime, the solution of the geodesic equation approaches a straight line in spacetime, which is the solution in SR.

2 The geodesic equation – 2

Here, we obtain the geodesic equation from the principle of least action applied to a free particle (freely falling along a geodesic).

2.1 Free-particle Lagrangian

The action must be a scalar invariant, and the only such scalar for a free particle is

$$S = -\alpha \int_a^b ds \tag{5}$$

where a, b are the initial and final position of the particle in spacetime, and α is a constant pertaining to the particle under study.

In the rest system of the particle, the spacetime interval is proportional to the proper time interval $ds = cd\tau$. If dt' is the time interval in the reference frame of the observer, then we can write the spacetime interval in both systems as follows

$$ds^2 = c^2dt^2 - (dx^2 + dy^2 + dz^2) = c^2d\tau^2 \tag{6}$$

where (x, y, z) denotes the position of the particle in the observer's reference frame, therefore

$$dt = \gamma d\tau \geq d\tau. \tag{7}$$

Writing the action in terms of Lagrangian, we find an equivalent expression for the integral in the observer's reference frame

$$S = -\alpha \int_{t_1}^{t_2} ds = -\alpha \int_{t_1}^{t_2} cd\tau = -\alpha \int_{t_1}^{t_2} c\sqrt{1-\beta^2}dt = \int_{t_1}^{t_2} Ldt \tag{8}$$

Expanding the square root at low β and using the Lagrangian for the free particle in Newtonian mechanics $L = mv^2/2$, we find that the last equality becomes

$$-\alpha c\left(1 - \frac{1}{2}\beta^2\right) \sim \frac{1}{2}mv^2 \tag{9}$$

and dropping the constant which is nonrelevant in setting up the Lagrange equations, we find

$$\alpha\frac{v^2}{2c} = \frac{1}{2}mv^2 \tag{10}$$

which shows that, in order to have agreement between the relativistic expression and the classical expression, we must have $\alpha = mc$, and that the expression for the action of the free particle is

$$S = -mc \int_a^b \sqrt{g_{\mu\nu}dx^\mu dx^\nu} \tag{11}$$

2.2 The geodesic equation from the Euler-Lagrange equations

Recall that, given a Lagrangian function $L(q_i, \dot{q}_i)$, the equations of motion can be obtained from the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{12}$$

In this case we work with a reduced Lagrangian, where we drop the multiplicative constants that play no role in the Lagrange equations

$$L(x^\alpha, \dot{x}^\alpha) = \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}} = \frac{d\tau}{d\sigma} \tag{13}$$

which is calculated along a trajectory parameterized by the scalar parameter σ, where the derivatives are computed with respect to this parameter, $\dot{x}^\mu = dx^\mu/d\sigma$ and

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{dx^\alpha} = 0. \tag{14}$$

In 4-dimensional space this corresponds to 4 equations in all.

The metric tensor changes along the curve, it only depends on x^μ and is not dependent on $\dot{x}^\mu$. Therefore

$$\frac{\partial L}{\partial \dot{x}^\alpha} = \frac{1}{2\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\left(g_{\mu\nu}\delta^\mu_\alpha\dot{x}^\nu + g_{\mu\nu}\dot{x}^\mu\delta^\nu_\alpha\right) = \frac{1}{2L}\left(g_{\alpha\nu}\dot{x}^\nu + g_{\mu\alpha}\dot{x}^\mu\right) = \frac{1}{L}g_{\alpha\mu}\frac{dx^\mu}{d\sigma} = g_{\alpha\mu}\frac{dx^\mu}{d\tau} \tag{15}$$

Similarly, we find

$$\frac{\partial L}{\partial x^\alpha} = \frac{1}{2L}\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau}. \tag{16}$$

Plugging these results in the Euler-Lagrange equations, we obtain

$$\frac{d}{d\sigma}\left(g_{\alpha\mu}\frac{dx^\mu}{d\tau}\right) - \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\tau} = 0, \tag{17}$$

and after multiplying times $d\sigma/d\tau$

$$\frac{d}{d\tau}\left(g_{\alpha\mu}\frac{dx^\mu}{d\tau}\right) - \frac{1}{2}\partial_\alpha g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0, \tag{18}$$

which is the geodesic equation obtained from the Lagrangian formulation.

3 Equivalence of the two geodesic equations

We have found two apparently different geodesic equations; those found with the method of parallel transport of the tangent vector are

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0, \tag{19}$$

and the geodesic equations obtained from the relativistic Lagrangian of a free particle are

$$\frac{d}{d\tau}\left(g_{\alpha\mu}\frac{dx^\mu}{d\tau}\right) - \frac{1}{2}\partial_\alpha g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0; \tag{20}$$

In this section we show that they are equivalent.

Notice first that

$$\frac{d}{d\tau}\left(g_{\alpha\mu}\frac{dx^{\mu}}{d\tau}\right)=\frac{dg_{\alpha\mu}}{d\tau}\frac{dx^{\mu}}{d\tau}+g_{\alpha\mu}\frac{dx^{\mu}}{d\tau} \tag{21}$$

$$=\partial_{\nu}g_{\alpha\mu}\dot{x}^{\mu}\dot{x}^{\nu}+g_{\alpha\mu}\frac{d^2x^{\mu}}{d\tau^2}, \tag{22}$$

therefore

$$\partial_{\nu}g_{\alpha\mu}\dot{x}^{\mu}\dot{x}^{\nu}+g_{\alpha\mu}\frac{d^2x^{\mu}}{d\tau^2}-\frac{1}{2}\partial_{\alpha}g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}=0. \tag{23}$$

Next, we exchange μ and α

$$\partial_{\nu}g_{\alpha\mu}\dot{x}^{\alpha}\dot{x}^{\nu}+g_{\alpha\mu}\frac{d^2x^{\alpha}}{d\tau^2}-\frac{1}{2}\partial_{\mu}g_{\alpha\nu}\dot{x}^{\alpha}\dot{x}^{\nu}=0, \tag{24}$$

rearrange the equation

$$\frac{d^2x^{\beta}}{d\tau^2}+\frac{1}{2}g^{\beta\mu}\left(\partial_{\nu}g_{\alpha\mu}+\partial_{\nu}g_{\alpha\mu}-\partial_{\mu}g_{\alpha\nu}\right)\dot{x}^{\alpha}\dot{x}^{\nu}=0, \tag{25}$$

exchange β and α

$$\frac{d^2x^{\alpha}}{d\tau^2}+\frac{1}{2}g^{\alpha\mu}\left(\partial_{\nu}g_{\beta\mu}+\partial_{\nu}g_{\beta\mu}-\partial_{\mu}g_{\beta\nu}\right)\dot{x}^{\beta}\dot{x}^{\nu}=0, \tag{26}$$

and finally, β and μ

$$\frac{d^2x^{\alpha}}{d\tau^2}+\frac{1}{2}g^{\alpha\beta}\left(\partial_{\nu}g_{\beta\mu}+\partial_{\nu}g_{\beta\mu}-\partial_{\beta}g_{\mu\nu}\right)\dot{x}^{\mu}\dot{x}^{\nu}=0 \tag{27}$$

Expanding the parenthesis, the second term is $\partial_{\nu}g_{\beta\mu}\dot{x}^{\mu}\dot{x}^{\nu}=\partial_{\mu}g_{\beta\nu}\dot{x}^{\mu}\dot{x}^{\nu}$ (exchanging μ and ν), and therefore

$$\frac{d^2x^{\alpha}}{d\tau^2}+\frac{1}{2}g^{\alpha\beta}\left(\partial_{\nu}g_{\beta\mu}+\partial_{\mu}g_{\beta\nu}-\partial_{\beta}g_{\mu\nu}\right)\dot{x}^{\mu}\dot{x}^{\nu}=0, \tag{28}$$

and recalling that

$$\Gamma^{i}_{jk}=\frac{1}{2}g^{i\ell}\left(\frac{\partial g_{\ell k}}{\partial x^{j}}+\frac{\partial g_{\ell j}}{\partial x^{k}}-\frac{\partial g_{jk}}{\partial x^{\ell}}\right)$$

we find the first version of the geodesic equation

$$\frac{d^2x^{\alpha}}{d\tau^2}+\frac{1}{2}\Gamma^{\alpha}_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}=0, \tag{29}$$

and conclude the proof of the equivalence.