The Riemann tensor

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1 Introduction

We transport a vector by keeping it parallel as we move from one point to a close one on a manifold. Parallel transport in flat space is trivial, as we do it following a closed loop, the transported vector overlaps the initial one when we close the loop (left panel of Fig. 1). The situation is much less trivial on a manifold with curvature, the right panel of Fig. 1 illustrates what happens on the surface of a sphere, we see that the transported vector no longer overlaps the initial one. Moreover, in curved space the final result depends on the path taken. This last remark has a stunning consequence, that in a curved spacetime we cannot really "compare velocities" at different spacetime points, because a comparison means that one of the two vectors must be "parallel transported" to the other one and this depends on the path.

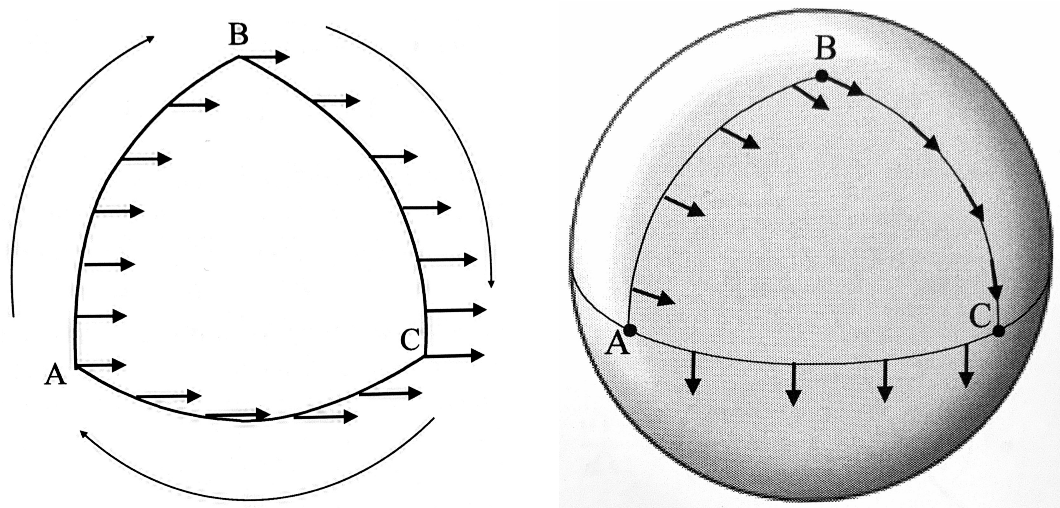

Figure 1: Left panel: parallel transport of a vector along a closed loop in flat space. Right panel: parallel transport of a vector along a closed loop on a sphere.

Parallel transport is closely connected to the curvature of a Riemann manifold. In this handout we explore these concepts.

2 Equation of parallel transport

Parallel transport generalizes to curved space the idea that we keep a vector constant as we move along a path $x^\mu(\sigma)$. For a generic tensor in flat space this means

$$\frac{d}{d\sigma} T^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_\ell} = \frac{dx^\mu}{d\sigma}\,\frac{\partial}{\partial x^\mu} T^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_\ell} = 0 \tag{1}$$

and we generalize the idea to curved spaces by replacing the partial derivative with the **directional covariant derivative**

$$\frac{D}{d\sigma} = \frac{dx^\mu}{d\sigma}\,\nabla_\mu \tag{2}$$

so that the equation of parallel transport of a tensor becomes

$$\frac{D}{d\sigma} T^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_\ell} = \frac{dx^\mu}{d\sigma}\,\nabla_\mu T^{\mu_1,\ldots,\mu_k}_{\nu_1,\ldots,\nu_\ell} = 0 \tag{3}$$

In the case of a contravariant vector, the equation becomes

$$\frac{dx^\mu}{d\sigma}\,\nabla_\mu V^\alpha = \frac{dx^\mu}{d\sigma}\left(\frac{\partial V^\alpha}{\partial x^\mu} + \Gamma^\alpha_{\mu\beta} V^\beta\right) = \frac{dV^\alpha}{d\sigma} + \Gamma^\alpha_{\mu\beta}\frac{dx^\mu}{d\sigma} V^\beta = 0 \tag{4}$$

2.1 Parallel transport of the metric tensor

The metric tensor always satisfies the parallel transport equation. This can be seen as follows:

$$\begin{aligned}
\nabla_\sigma g_{\mu\nu} &= \partial_\sigma g_{\mu\nu} - g_{\lambda\nu}\Gamma^\lambda_{\mu\sigma} - g_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} \\
&= \partial_\sigma g_{\mu\nu} - g_{\lambda\nu}\frac{1}{2} g^{\lambda\alpha}\left(\partial_\mu g_{\alpha\sigma} + \partial_\sigma g_{\mu\alpha} - \partial_\alpha g_{\mu\sigma}\right) - g_{\mu\lambda}\frac{1}{2} g^{\lambda\alpha}\left(\partial_\nu g_{\alpha\sigma} + \partial_\sigma g_{\alpha\nu} - \partial_\alpha g_{\sigma\nu}\right) \\
&= \partial_\sigma g_{\mu\nu} - \frac{1}{2}\delta^\alpha_\nu\left(\partial_\mu g_{\alpha\sigma} + \partial_\sigma g_{\mu\alpha} - \partial_\alpha g_{\mu\sigma}\right) - \frac{1}{2}\delta^\alpha_\mu\left(\partial_\nu g_{\alpha\sigma} + \partial_\sigma g_{\alpha\nu} - \partial_\alpha g_{\sigma\nu}\right) \\
&= \partial_\sigma g_{\mu\nu} - \frac{1}{2}\left(\partial_\mu g_{\nu\sigma} + \partial_\sigma g_{\mu\nu} - \partial_\nu g_{\mu\sigma}\right) - \frac{1}{2}\left(\partial_\nu g_{\mu\sigma} + \partial_\sigma g_{\mu\nu} - \partial_\mu g_{\sigma\nu}\right) \\
&= 0
\end{aligned} \tag{5}$$

This means that

$$\frac{D}{d\sigma} g_{\alpha\beta} = \frac{dx^\mu}{d\sigma}\,\nabla_\mu g_{\alpha\beta} = 0 \tag{6}$$

and **the metric tensor always satisfies the parallel transport equation**.

2.2 Parallel transport of the inner product of two parallel-transported vectors

The inner product of two contravariant parallel-transported vectors is defined by $g_{\mu\nu}A^\mu B^\nu$, therefore

$$\frac{D}{d\sigma}\left(g_{\mu\nu}A^\mu B^\nu\right) = \left(\frac{D}{d\sigma}g_{\mu\nu}\right)A^\mu B^\nu + g_{\mu\nu}\left(\frac{D}{d\sigma}A^\mu\right)B^\nu + g_{\mu\nu}A^\mu\left(\frac{D}{d\sigma}B^\nu\right) = 0 \qquad (7)$$

because each term vanishes. This means that **parallel transport conserves the norm of parallel-transported vectors, their orthogonality, etc.**

2.3 The geodesic equation as a consequence of parallel transport

The tangent vector to the path $x^\mu(\sigma)$ is $dx^\mu/d\sigma$, therefore the equation for parallel transport is just

$$\frac{d^2x^\alpha}{d\sigma^2} + \Gamma^\alpha_{\mu\beta}\frac{dx^\mu}{d\sigma}\frac{dx^\beta}{d\sigma} = 0 \qquad (8)$$

which is the geodesic equation.

2.4 The local flatness theorem

Consider a transformation to primed coordinates that transforms the metric tensor as follows

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}g_{\alpha\beta}, \qquad (9)$$

here we prove that it is possible to find a transformation such that $g'_{\mu\nu} = \eta_{\mu\nu}$ and $\partial'_\alpha g'_{\mu\nu} = 0$.

We start by series expanding the 16 elements of the Jacobian matrix at a spacetime point P, relative to the primed coordinates

$$\frac{\partial x^\alpha}{\partial x'^\mu} \approx a^\alpha_\mu + b^\alpha_{\mu\lambda}\Delta x'^\lambda + c^\alpha_{\mu\lambda\sigma}\Delta x'^\lambda \Delta x'^\sigma \qquad (10)$$

where

$$\Delta x'^\mu = x'^\mu - [x'^\mu]_P \qquad (11)$$

$$a^\alpha_\mu = \left[\frac{\partial x^\alpha}{\partial x'^\mu}\right]_P \qquad (12)$$

$$b^\alpha_{\mu\lambda} = \left[\frac{\partial}{\partial x'^\lambda}\left(\frac{\partial x^\alpha}{\partial x'^\mu}\right)\right]_P = \left[\frac{\partial^2 x^\alpha}{\partial x'^\lambda \partial x'^\mu}\right]_P \qquad (13)$$

$$c^\alpha_{\mu\lambda\sigma} = \frac{1}{2}\left[\frac{\partial^2}{\partial x'^\lambda \partial x'^\sigma}\left(\frac{\partial x^\alpha}{\partial x'^\mu}\right)\right]_P = \frac{1}{2}\left[\frac{\partial^3 x^\alpha}{\partial x'^\lambda \partial x'^\sigma \partial x'^\mu}\right]_P \qquad (14)$$

The choice of the coordinate transformation is free, and therefore we are free to choose the a, b and c coefficients: we choose them in such a way that $g'_{\mu\nu} = \eta_{\mu\nu}$ and $\partial'_\alpha g'_{\mu\nu} = 0$.

To start with, we count the number of independent values for each set of coefficients:

- there are no symmetries in the indexes of the a^α_μs, therefore here we have 16 independent values,
- the $b^\alpha_{\mu\lambda}$s have an obvious exchange symmetry between the covariant indexes. This means that there are 40 independent values,
- again, the $c^\alpha_{\mu\lambda\sigma}$s have 80 distinct combination of indexes.

Next, we expand $g_{\alpha\beta}$ in the primed coordinate system

$$g_{\alpha\beta} = [g_{\alpha\beta}]_P + \left[\partial'_\gamma g_{\alpha\beta}\right]_P \Delta x'^\gamma + \frac{1}{2}\left[\partial'_\gamma \partial'_\delta g_{\alpha\beta}\right]_P \Delta x'^\gamma \Delta x'^\delta \tag{15}$$

This means that we can write

$$\begin{aligned} g'_{\mu\nu} \approx & \left(a^\alpha_\mu + b^\alpha_{\mu\lambda_1}\Delta x'^{\lambda_1} + c^\alpha_{\mu\lambda_1\sigma_1}\Delta x'^{\lambda_1}\Delta x'^{\sigma_1}\right) \\ & \times \left(a^\beta_\nu + b^\beta_{\nu\lambda_2}\Delta x'^{\lambda_2} + c^\beta_{\nu\lambda_2\sigma_2}\Delta x'^{\lambda_2}\Delta x'^{\sigma_2}\right) \\ & \times \left([g_{\alpha\beta}]_P + \left[\partial'_\gamma g_{\alpha\beta}\right]_P \Delta x'^\gamma + \frac{1}{2}\left[\partial'_\gamma \partial'_\delta g_{\alpha\beta}\right]_P \Delta x'^\gamma \Delta x'^\delta\right) \\ \approx & \, a^\alpha_\mu a^\beta_\nu [g_{\alpha\beta}]_P + \left(a^\beta_\nu b^\alpha_{\mu\gamma}[g_{\alpha\beta}]_P + a^\alpha_\mu b^\beta_{\nu\gamma}[g_{\alpha\beta}]_P + a^\alpha_\mu a^\beta_\nu \left[\partial'_\gamma g_{\alpha\beta}\right]_P\right)\Delta x'^\gamma \\ & + \left(a^\alpha_\mu c^\beta_{\nu\gamma\delta}[g_{\alpha\beta}]_P + a^\beta_\nu c^\alpha_{\mu\gamma\delta}[g_{\alpha\beta}]_P + \text{terms with } a \text{ and } b \text{ only}\right)\Delta x'^\gamma \Delta x'^\delta \end{aligned} \tag{16}$$

Now consider the first term in the r.h.s. of Eq. (16); it depends on a alone, and it determines the value of $g'_{\mu\nu}$ at the spacetime point P. Since $g'_{\mu\nu}$ has only 10 independent elements, we see that the degrees of freedom carried by a are more than enough to let us choose a in such a way that $[g'_{\mu\nu}]_P = \eta_{\mu\nu}$. The 6 remaining degrees of freedom can be used for local rotations and Lorentz boosts.

The next question is, is this flatness stable enough to guarantee that in a neighbourhood of P the metric tensor does not change much and we can effectively say that this neighbourhood is locally flat? The $\Delta x'^\gamma$-dependent term can be tuned with the b coefficients, and it should yield the $\left[\partial'_\gamma g'_{\alpha\beta}\right]_P$ term in the expansion of $g'_{\alpha\beta}$. Since this term corresponds to 40 independent coefficients, we have just enough freedom in the choice of coefficients to guarantee that we can set $\left[\partial'_\gamma g'_{\alpha\beta}\right]_P = 0$. **This means that we can choose coordinates such that the coordinate frame appears as locally flat in a small neighbourhood, i.e., a Local Inertially Frame (LIF).**

Can we play the same trick with local curvature (represented by the next term, dependent on $\Delta x'^\gamma \Delta x'^\delta$)? At this point we have exhausted the degrees of freedom provided by a and b and we are left with just c (80 degrees of freedom). This is used to determine the next term in the expansion of $g'_{\mu\nu}$, which corresponds to 100 degrees of freedom, and we see that now we do not have enough freedom to choose a coordinate system that leads to a local vanishing of curvature.

Finally, we find that the existence of the LIF is fully compatible with the earlier equality (5), which is also related with the symmetry of the Christoffel symbols: these results are all strictly interrelated.

3 The Riemann tensor

In the first section we noticed that curvature produces a mismatch in the parallel transport of a vector around a loop. Clearly, to characterize local curvature, we should consider small, infinitesimal loops. Moreover, it is not strictly necessary to get back at

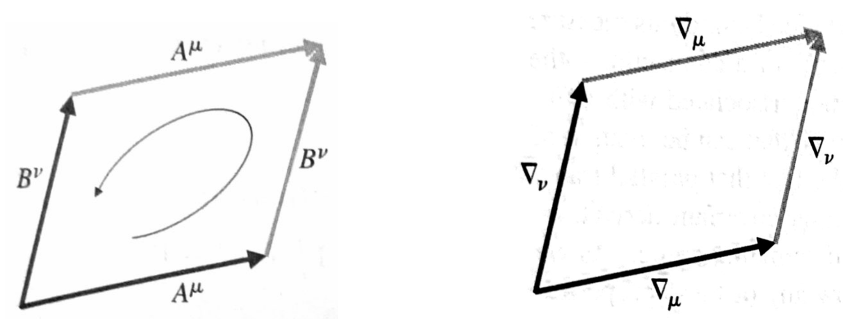

Figure 2: Left panel: a small loop defined by two (small) vectors A and B. Right panel: operators associated with parallel transport along the loop sides.

the origin of the loop, we could also move along "parallelogram edges" like those in the left panel of figure 2: the parallelogram is equivalent to a loop, but it lets us clearly display the difference between apparently equivalent paths to reach the same endpoint. We write this difference as follows

$$\delta V^\rho = R^\rho_{\sigma\mu\nu} V^\sigma A^\mu B^\nu \tag{17}$$

because in a small loop we expect the difference to be proportional to the vector size, and to the distance over which we transport the vector (i.e., along A and B). The set of proportionality constants $R^\rho_{\sigma\mu\nu}$ also depend on the directions of these vectors. Because of the quotient rule, $R^\rho_{\sigma\mu\nu}$ is itself a tensor, known as the **Riemann tensor**. Changing the direction of the loop, the difference must change sign, therefore $R^\rho_{\sigma\mu\nu}$ must be antisymmetric with respect to μ and ν.

More specifically, from the discussion in the previous section, we see that the covariant derivative is the proper operator to characterize differences in the parallel transport of a vector any given direction on the manifold, and the order with which we take parallel transport defines a commutator (left panel in figure 2). Here, we expand the commutator

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu]V^\rho &= \nabla_\mu \nabla_\nu V^\rho - \nabla_\nu \nabla_\mu V^\rho && (18)\\
&= \partial_\mu(\nabla_\nu V^\rho) + \Gamma^\rho_{\mu\lambda}\nabla_\nu V^\lambda - \Gamma^\sigma_{\mu\nu}\nabla_\sigma V^\rho - (\mu \leftrightarrow \nu) && (19)\\
&= \partial_\mu(\partial_\nu V^\rho + \Gamma^\rho_{\nu\lambda}V^\lambda) + \Gamma^\rho_{\mu\lambda}(\partial_\nu V^\lambda + \Gamma^\lambda_{\nu\sigma}V^\sigma) - \Gamma^\sigma_{\mu\nu}(\partial_\sigma V^\rho + \Gamma^\rho_{\sigma\lambda}V^\lambda) - (\mu \leftrightarrow \nu) && (20)\\
&= \partial_\mu\partial_\nu V^\rho + (\partial_\mu \Gamma^\rho_{\nu\lambda})V^\lambda + \Gamma^\rho_{\nu\lambda}\partial_\mu V^\lambda + \Gamma^\rho_{\mu\lambda}\partial_\nu V^\lambda + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}V^\sigma \\
&\quad - \Gamma^\sigma_{\mu\nu}\partial_\sigma V^\rho - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\lambda}V^\lambda \\
&\quad - \Big(\partial_\nu\partial_\mu V^\rho + (\partial_\nu \Gamma^\rho_{\mu\lambda})V^\lambda + \Gamma^\rho_{\mu\lambda}\partial_\nu V^\lambda + \Gamma^\rho_{\nu\lambda}\partial_\mu V^\lambda + \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}V^\sigma \\
&\quad - \Gamma^\sigma_{\mu\nu}\partial_\sigma V^\rho - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\lambda}V^\lambda\Big) && (21)\\
&= (\partial_\mu \Gamma^\rho_{\nu\lambda})V^\lambda - (\partial_\nu \Gamma^\rho_{\mu\lambda})V^\lambda + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}V^\sigma - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}V^\sigma && (22)\\
&= \Big(\partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda}\Big) V^\lambda && (23)
\end{aligned}
$$

(this derivation is given as in [1]). Thus, we see that

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho_{\lambda\mu\nu}V^\lambda \tag{24}$$

where the Riemann tensor is

$$R^\rho_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\lambda} - \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} - \Gamma^\rho_{\nu\sigma}\Gamma^\sigma_{\mu\lambda} \tag{25}$$

Moreover, it is quite obvious that the Riemann tensor is antisymmetric with respect to the last two indexes.

Notice also that

- If a coordinate system exists in which the components of the metric are constant, then the Riemann tensor vanishes
- If the Riemann tensor vanishes, then we can construct a coordinate system where the metric is constant.

4 Contractions of the Riemann tensor

There are two useful contractions of the Riemann tensor:

the Ricci tensor:

$$R_{\mu\nu} = R^\alpha_{\mu\nu\alpha} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\mu \Gamma^\alpha_{\alpha\nu} + \Gamma^\alpha_{\alpha\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\alpha\nu} \tag{26}$$

the Ricci scalar: this is the trace of the Ricci tensor

$$R = g^{\mu\nu}R_{\mu\nu} = R^\nu_\nu \tag{27}$$

References

[1] Sean M Carroll. *Spacetime and geometry.* Cambridge University Press, 2019.