Example: Ricci tensor and curvature of the surface of a 3D sphere

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ln this handout we compute the non–zero components of the Ricci tensor and the Ricci scalar for a simple 2D manifold, the surface of the 3D sphere with fixed radius r. In this case the Ricci tensor has 3 independent components.

Before starting, it is important to recall the following formulas:

1. The connection coefficients can be obtained from

$$\Gamma^n_{ij} = \frac{1}{2} g^{nk} \left(\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}\right) \tag{1}$$

2. In terms of connection coefficients, the Riemann tensor is

$$R^d_{abc} = \partial_b \Gamma^d_{ac} - \partial_c \Gamma^d_{ab} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{ab}\Gamma^d_{ce}, \tag{2}$$

3. The Ricci tensor is just the contraction

$$R_{ab} = R^d_{abd} = \partial_b \Gamma^d_{ad} - \partial_d \Gamma^d_{ab} + \Gamma^e_{ad}\Gamma^d_{be} - \Gamma^e_{ab}\Gamma^d_{de} \tag{3}$$

- The metric tensor. In the case of the sphere, we write:

$$x = r \sin\theta \cos\varphi \tag{4}$$
$$y = r \sin\theta \sin\varphi \tag{5}$$
$$z = r \cos\theta \tag{6}$$

Therefore

$$dx = r\cos\theta\cos\varphi d\theta - r\sin\theta\sin\varphi d\varphi \tag{7}$$
$$dy = r\cos\theta\sin\varphi d\theta + r\sin\theta\cos\varphi d\varphi \tag{8}$$
$$dz = -r\sin\theta d\theta \tag{9}$$

and

$$ds^2 = dx^2 + dz^2 + dy^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \tag{10}$$

so that

$$[g_{ij}] = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\theta \end{pmatrix}; \quad [g^{ij}] = \begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} \tag{11}$$

- Because of the symmetry of the covariant indices, the connection coefficients have 6 independent components, out of a total of 8 components, and we find

$$\Gamma^\theta_{\theta\theta} = 0; \quad \Gamma^\varphi_{\theta\theta} = 0$$
$$\Gamma^\theta_{\theta\varphi} = \Gamma^\theta_{\varphi\theta} = 0$$
$$\Gamma^\varphi_{\theta\varphi} = \cot\theta; \quad \Gamma^\varphi_{\varphi\theta} = \cot\theta$$
$$\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta; \quad \Gamma^\varphi_{\varphi\varphi} = 0$$

- Applying eq. (3) we find the components of the Ricci tensor

$$R_{\theta\theta} = \partial_\theta\Gamma^d_{\theta d} - \partial_d\Gamma^d_{\theta\theta} + \Gamma^k_{\theta d}\Gamma^d_{\theta k} - \Gamma^k_{\theta\theta}\Gamma^d_{dk} = \partial_\theta\Gamma^d_{\theta d} + \Gamma^k_{\theta d}\Gamma^d_{\theta k} = -1 \tag{12}$$
$$R_{\theta\varphi} = R_{\varphi\theta} = 0 \tag{13}$$
$$R_{\varphi\varphi} = -\sin^2\theta \tag{14}$$

- The Ricci scalar is the contraction of the Ricci tensor

$$R = g^{ij}R_{ij} = -\frac{2}{r^2} \tag{15}$$

The last result is noteworthy, because it reflects the more general behavior of the components of the Riemann and Ricci tensors: the larger components have magnitude of the order $1/r^2$, therefore the radius of curvature of the manifold has magnitude of the order of $\sqrt{1/|R|}$ where here R represents a generic (larger) component of the Riemann or Ricci tensor.

Notice also that a very large radius of curvature (manifold close to flat space) implies small components of the Riemann and Ricci tensors.