Indipendent components of the Riemann and Ricci tensors

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October 31, 2023

The Riemann curvature tensor has 4 indexes, therefore it has 256 components. However not all the components are different, and here we count the actual number of different components.

Recall the identities

$$R_{\mu\alpha\beta\gamma} = -R_{\alpha\mu\beta\gamma} = -R_{\mu\alpha\gamma\beta} = R_{\beta\gamma\mu\alpha}, \tag{1}$$

(the Riemann tensor is antisymmetric with respect to the exchange of both the first pair of indexes and the second pair of indexes, while it does not change in a double exchange $1 \leftrightarrow 3$, $2 \leftrightarrow 4$) and

$$R_{\mu\alpha\beta\gamma} + R_{\mu\beta\gamma\alpha} + R_{\mu\gamma\alpha\beta} = 0 \tag{2}$$

(the *cyclic identity*).

The first antisymmetry means that if $\alpha = \mu$ then $R_{\mu\alpha\beta\gamma} = 0$. In order to obtain a nonzero value there are 4 choices for the first index and 3 for the second one. Since order does not matter, there are in all 6 different pairs. The same holds for the second pair of indexes. This means that there are potentially 36 nonzero independent components.

However, the third equality $R_{\mu\alpha\beta\gamma} = R_{\beta\gamma\mu\alpha}$ means that one can exchange the first pair with the second pair without altering the value of the coefficient (in practice, the matrix of values indexed by the $\mu\alpha$ pair for the rows and the $\beta\gamma$ pair for the columns is symmetrical), and therefore there are at most $n(n+1)/2 = 6 \times 7/2 = 21$ independent values (recall that $n = 6$ is the number of rows and columns of this matrix).

We still have to use the last identity

$$R_{\mu\alpha\beta\gamma} + R_{\mu\beta\gamma\alpha} + R_{\mu\gamma\alpha\beta} = 0, \tag{3}$$

where we can raise the first index to find

$$R^{\mu}_{\alpha\beta\gamma} + R^{\mu}_{\beta\gamma\alpha} + R^{\mu}_{\gamma\alpha\beta} = 0. \tag{4}$$

However, it is easy to see that whenever two indexes are equal, this is identically null simply because of the previous identities (Exercise: verify this statement). Therefore, the only additional constraint is

$$R_{0123} + R_{0231} + R_{0312} = 0 \tag{5}$$

and this brings down to 20 the total number of independent components of the Riemann curvature tensor.

Now, note that by contracting the cyclic identity over μ and γ, we find

$$R^{\mu}_{\alpha\beta\mu} + R^{\mu}_{\beta\mu\alpha} + R^{\mu}_{\mu\alpha\beta} = 0. \tag{6}$$

Since the Riemann tensor is antisymmetric with respect to the exchange of the first pair of indices the last term of this identity vanishes and we are left with

$$R^{\mu}_{\alpha\beta\mu} + R^{\mu}_{\beta\mu\alpha} = 0, \tag{7}$$

and because of the antisymmetry of the last two indices $R^{\mu}_{\alpha\beta\mu} = -R^{\mu}_{\alpha\mu\beta}$ we find

$$R^{\mu}_{\alpha\mu\beta} = R^{\mu}_{\beta\mu\alpha}, \tag{8}$$

and finally, recalling the definition of the Ricci tensor

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \tag{9}$$

we find

$$R_{\alpha\beta} = R_{\beta\alpha}, \tag{10}$$

i.e., the Ricci tensor is a rank-2 symmetric tensor and therefore it has only 10 independent components