Geodesics in curved spacetime

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We start from the geodesic equations

$$\frac{d^2x^a}{du^2} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} = 0, \tag{1}$$

where the u is the curve parameter and $x^a = x^a(u)$ (latin indexes, because these equations hold true for any Riemann manifold). Next, we consider two neighboring geodesics identified by the coordinates $x^a(u)$ and $\tilde{x}^a(u)$ and their (small) difference – the **geodesic deviation – at given** u, $\xi^a(u) = \tilde{x}^a(u) - x^a(u)$ (see Fig 1).

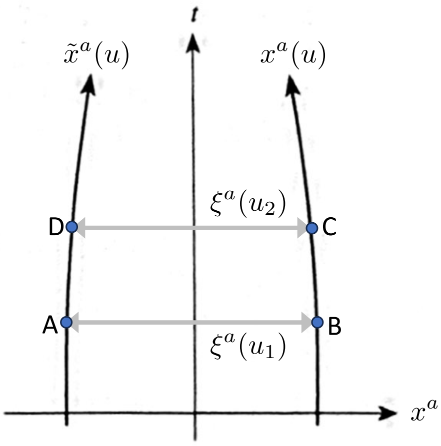

Figure 1: Two geodesics in spacetime with the geodesic deviation marked at two different values of u (proportional to proper time). The spacetime points A, B, C, and D mark a loop, therefore it is reasonable to expect that an equation that describes the geodesic deviation ξ^a has some connection with the Riemann tensor.

When we consider the geodesic equation for $\tilde{x}^a(u)$

$$\frac{d^2\tilde{x}^a}{du^2} + \tilde{\Gamma}^a_{bc}\frac{d\tilde{x}^b}{du}\frac{d\tilde{x}^c}{du} = 0, \tag{2}$$

where $\tilde{\Gamma}^a_{bc}$ is evaluated at $\tilde{x}^a(u) = x^a(u) + \xi^a(u)$, and subtract eq. (1) from eq. (2), we find

$$\frac{d^2\xi^a}{du^2} = \frac{d^2\tilde{x}^a}{du^2} - \frac{d^2x^a}{du^2} = -\left(\tilde{\Gamma}^a_{bc}\frac{d\tilde{x}^b}{du}\frac{d\tilde{x}^c}{du} - \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du}\right). \tag{3}$$

We expand the Christoffel symbol $\tilde{\Gamma}^a_{bc}$ to first order in ξ^a (the geodesic deviation is assumed to be small) as follows:

$$\tilde{\Gamma}^a_{bc} \approx \Gamma^a_{bc} + \partial_d\Gamma^a_{bc}\xi^d, \tag{4}$$

then, substituting in eq. (3) and neglecting terms that are superlinear in ξ and its derivatives, we find

$$\begin{aligned}
\frac{d^2\xi^a}{du^2} &= -\left(\Gamma^a_{bc} + \partial_d\Gamma^a_{bc}\,\xi^d\right)\frac{d(x^b+\xi^b)}{du}\frac{d(x^c+\xi^c)}{du} + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} && (5)\\
&\approx -\left(\Gamma^a_{bc} + \partial_d\Gamma^a_{bc}\,\xi^d\right)\left[\frac{dx^b}{du}\frac{dx^c}{du} + \dot{\xi}^b\frac{dx^c}{du} + \dot{\xi}^c\frac{dx^b}{du}\right] + \Gamma^a_{bc}\frac{dx^b}{du}\frac{dx^c}{du} && (6)\\
&\approx -\Gamma^a_{bc}\dot{\xi}^b\frac{dx^c}{du} - \Gamma^a_{bc}\dot{\xi}^c\frac{dx^b}{du} - \partial_d\Gamma^a_{bc}\,\xi^d\frac{dx^b}{du}\frac{dx^c}{du} && (7)\\
&\approx -\Gamma^a_{bc}\dot{\xi}^b\dot{x}^c - \Gamma^a_{bc}\dot{\xi}^c\dot{x}^b - \partial_d\Gamma^a_{bc}\,\xi^d\dot{x}^b\dot{x}^c && (8)
\end{aligned}$$

Next, adding to both sides the expression

$$\frac{d}{du}\Gamma^a_{bc}\xi^b\dot{x}^c = \left(\frac{d}{du}\Gamma^a_{bc}\right)\xi^b\dot{x}^c + \Gamma^a_{bc}\dot{\xi}^b\dot{x}^c + \Gamma^a_{bc}\xi^b\ddot{x}^c = \partial_d\Gamma^a_{bc}\xi^b\dot{x}^d\dot{x}^c + \Gamma^a_{bc}\dot{\xi}^b\dot{x}^c + \Gamma^a_{bc}\xi^b\ddot{x}^c \tag{9}$$

where we use the chain rule, Eq. (8) can be rearranged as follows:

$$\frac{d}{du}\left(\dot{\xi}^a + \Gamma^a_{bc}\xi^b\dot{x}^c\right) - \partial_d\Gamma^a_{bc}\,\xi^b\dot{x}^c\dot{x}^d - \Gamma^a_{bc}\xi^b\ddot{x}^c + \Gamma^a_{bc}\dot{\xi}^c\dot{x}^b + \partial_d\Gamma^a_{bc}\,\xi^d\dot{x}^b\dot{x}^c = 0. \tag{10}$$

Now notice that the geodesic equation (1) can also be written in the form

$$\ddot{x}^c = -\Gamma^c_{de}\dot{x}^d\dot{x}^e \tag{11}$$

which holds on the geodesic: we use this equation to get rid of the second derivative $\ddot{x}^c$, so that eq. (10) becomes

$$\frac{d}{du}\left(\dot{\xi}^a + \Gamma^a_{bc}\xi^b\dot{x}^c\right) - \partial_d\Gamma^a_{bc}\,\xi^b\dot{x}^c\dot{x}^d + \Gamma^a_{bc}\Gamma^c_{de}\xi^b\dot{x}^d\dot{x}^e + \Gamma^a_{bc}\dot{\xi}^c\dot{x}^b + \partial_d\Gamma^a_{bc}\,\xi^d\dot{x}^b\dot{x}^c = 0. \tag{12}$$

We recall here the definition of the absolute derivative

$$\frac{Dv^a}{du} = \dot{v}^a + \Gamma^a_{bc}v^b\dot{x}^c \tag{13}$$

and we remark that the term enclosed in the parenthesis on the l.h.s. of Eq. (12) is the absolute derivative of ξ. We also note that

$$\frac{d}{du}\left(\dot{\xi}^a + \Gamma^a_{bc}\xi^b\dot{x}^c\right) = \frac{D}{du}\left(\dot{\xi}^a + \Gamma^a_{bc}\xi^b\dot{x}^c\right) - \Gamma^a_{de}\left(\dot{\xi}^d + \Gamma^d_{bc}\xi^b\dot{x}^c\right)\dot{x}^e \tag{14}$$

$$= \frac{D^2\xi^a}{du^2} - \Gamma^a_{de}\left(\dot{\xi}^d + \Gamma^d_{bc}\xi^b\dot{x}^c\right)\dot{x}^e. \tag{15}$$

With this rearrangement of the first term, eq. (12) becomes

$$\frac{D^2\xi^a}{du^2} - \Gamma^a_{de}\left(\dot{\xi}^d + \Gamma^d_{bc}\xi^b\dot{x}^c\right)\dot{x}^e - \partial_d\Gamma^a_{bc}\,\xi^b\dot{x}^c\dot{x}^d + \Gamma^a_{bc}\Gamma^c_{de}\xi^b\dot{x}^d\dot{x}^e + \Gamma^a_{bc}\dot{\xi}^c\dot{x}^b + \partial_d\Gamma^a_{bc}\,\xi^d\dot{x}^b\dot{x}^c = 0, \tag{16}$$

or also

$$\frac{D^2\xi^a}{du^2} - \Gamma^a_{de}\dot{\xi}^d\dot{x}^e - \Gamma^a_{de}\Gamma^d_{bc}\xi^b\dot{x}^c\dot{x}^e - \partial_d\Gamma^a_{bc}\,\xi^b\dot{x}^c\dot{x}^d + \Gamma^a_{bc}\Gamma^c_{de}\xi^b\dot{x}^d\dot{x}^e + \Gamma^a_{bc}\dot{\xi}^c\dot{x}^b + \partial_d\Gamma^a_{bc}\,\xi^d\dot{x}^b\dot{x}^c = 0. \tag{17}$$

Taking into account the symmetry of the lower indexes of the Christoffel symbols and with the proper substitutions of indexes in the contractions, we see that the second and the sixth term in eq. (17) (marked in red) have the same value and opposite sign, and eq. (17) simplifies to

$$\frac{D^2\xi^a}{du^2} - \Gamma^a_{de}\Gamma^d_{bc}\xi^b\dot{x}^c\dot{x}^e - \partial_d\Gamma^a_{bc}\,\xi^b\dot{x}^c\dot{x}^d + \Gamma^a_{bc}\Gamma^c_{de}\xi^b\dot{x}^d\dot{x}^e + \partial_d\Gamma^a_{bc}\,\xi^d\dot{x}^b\dot{x}^c = 0, \tag{18}$$

and after rearranging the terms and changing the names of indices in contractions

$$\frac{D^2\xi^a}{du^2} + \left(\partial_b\Gamma^a_{de}\,\xi^b\dot{x}^d\dot{x}^e - \partial_d\Gamma^a_{be}\,\xi^b\dot{x}^d\dot{x}^e + \Gamma^a_{bc}\Gamma^c_{de}\xi^b\dot{x}^d\dot{x}^e - \Gamma^a_{ce}\Gamma^c_{bd}\xi^b\dot{x}^d\dot{x}^e\right) = 0, \tag{19}$$

i.e.,

$$\frac{D^2\xi^a}{du^2} + \left(\partial_b\Gamma^a_{de} - \partial_d\Gamma^a_{be} + \Gamma^a_{bc}\Gamma^c_{de} - \Gamma^a_{ce}\Gamma^c_{bd}\right)\xi^b\dot{x}^d\dot{x}^e = 0 \tag{20}$$

Recalling the definition of the Riemann curvature tensor

$$R^d_{abc} = \partial_b\Gamma^d_{ac} - \partial_c\Gamma^d_{ab} + \Gamma^e_{ac}\Gamma^d_{be} - \Gamma^e_{ab}\Gamma^d_{ce}, \tag{21}$$

we see that eq. (20) reduces to

$$\frac{D^2\xi^a}{du^2} + R^a_{dbe}\xi^b\dot{x}^d\dot{x}^e = 0. \tag{22}$$

which is the **equation of geodesic deviation**.