The stress-energy (energy-momentum) tensor

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January 26, 2024

One intuition of Einstein's was that – like in electromagnetic case – the equations of the curvature field should include a source term. Since the idea is that curvature is related to the presence of energy, the source term should express this concept.

Here, I stress that the source of the gravitational field must be energy, not mass. This can be shown with a simple thought experiment, where a box holds in separate positions a particle and an antiparticle. Next, another particle is released from a distance, and as the particle gains speed by consuming gravitational potential energy, it finally passes by the box. At this time particle and antiparticle in the box are set free and annihilate, and we are left only with the annihilation photons – and no mass. If mass – and not energy – were the source of the gravitational field, this would mean that the passing particle could fly away with all the kinetic energy gained during the fall and thereby violate the conservation of energy.

1 The stress-energy tensor

The classical expression of the conservation equation for the number density $n(\mathbf{t}, \mathbf{x})$ and associated current $n\mathbf{v}$ is

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0. \tag{1}$$

When one considers a Lorentz boost that compresses volumes in the direction of motion, the observed densities are proportionally higher. The following modified form takes into account the Lorentz boost and introduces $x^0 = ct$

$$\frac{\partial nc\gamma}{\partial(ct)} + \nabla \cdot (n\gamma\mathbf{v}) = 0. \tag{2}$$

The latter formula shows that one can define a 4-current $J^\mu = nU^\mu$, where U is the 4-velocity $(\gamma c, \gamma\mathbf{v})$, and obtain the SR continuity equation

$$\partial_\alpha(nU^\alpha) = 0. \tag{3}$$

Here, I have used the number density $n(\mathbf{t}, \mathbf{x})$, but the corresponding matter density for particles having each a mass m is $\rho_0 = mn(\mathbf{t}, \mathbf{x})$ which, when subject to a Lorentz boost, becomes $\rho = \gamma^2 mn(\mathbf{t}, \mathbf{x}) = \rho_0\gamma^2$, which transforms as the 0 component of a rank-2 tensor

(ρ_0 is the proper density of the dust of particle as measured by a comoving observer). This hints that a relativistic extension should be bilinear, i.e., it should be a rank-2 tensor. Another hint is that J^μ is linear with respect to speed, but an extension that includes the kinetic energy should be quadratic. An obvious way to obtain the required tensor is the following tensor product

$$T^{\mu\nu} = \rho U^\mu U^\nu. \tag{4}$$

From the definition (4) we see that tensor satisfies the continuity equation

$$\partial_\mu T^{\mu\nu} = 0, \tag{5}$$

and that can it be represented in matrix form in a given LIF

$$[T^{\mu\nu}] = \begin{pmatrix} \rho\gamma^2 c^2 & \rho\gamma^2 cv^1 & \rho\gamma^2 cv^2 & \rho\gamma^2 cv^3 \\ \rho\gamma^2 cv^1 & \rho\gamma^2 v^1 v^1 & \rho\gamma^2 v^1 v^2 & \rho\gamma^2 v^1 v^3 \\ \rho\gamma^2 cv^2 & \rho\gamma^2 v^1 v^2 & \rho\gamma^2 v^2 v^2 & \rho\gamma^2 v^2 v^3 \\ \rho\gamma^2 cv^3 & \rho\gamma^2 v^1 v^3 & \rho\gamma^2 v^2 v^3 & \rho\gamma^2 v^3 v^3 \end{pmatrix} \tag{6}$$

with the associated meanings:

- T^{00} is the energy density of the particles,
- the components $T^{0i} = T^{i0} = \rho\gamma^2 cv^i = \frac{1}{c}\rho c^2\gamma^2 v^i$ are the energy flux in the i-th direction, divided by c, or, equivalently, the momentum density in the i-direction times c,
- the components $T^{ij} = \rho\gamma^2 v^i v^j$ represent the flow of the i-th component of momentum per unit area per unit time in the j-th direction.

These correspondences are given here for **dust**, but are retained for fluids, where they are interpreted accordingly:

- T^{00} is the total energy density of the fluid, including potential energy terms and kinetic energy from internal random motions,
- the components $T^{0i} = T^{i0} = \rho\gamma^2 cv^i = \frac{1}{c}\rho c^2\gamma^2 v^i$ are the energy flux in the i-th direction, divided by c, or, equivalently, the momentum density in the i-direction times c; in the absence of actual motion these terms may represent heat transport by conduction,
- since momentum per unit area per unit time is just the fluid pressure, we can interpret the T^{ii} components as fluid pressure in the i-th direction and the remaining T^{ij} components as the viscous stresses of the fluid represent the flow of the i-th component of momentum per unit area per unit time in the j-th direction.

Thus, we see that the stress-energy tensor for a perfect fluid (with vanishing viscosity) in its rest frame can be written as

$$[T^{\mu\nu}] = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{7}$$

In order to find a formula valid for all frames, one can guess an extension of the initial definition

$$T^{\mu\nu} = \rho U^\mu U^\nu \quad \longrightarrow \quad (\rho + p/c^2) U^\mu U^\nu. \tag{8}$$

Unfortunately, this does not return T^{00} in the rest frame, but the problem is easily fixed by subtracting the tensor $p\eta^{\mu\nu}$:

$$T^{\mu\nu} = (\rho + p/c^2) U^\mu U^\nu - p\eta^{\mu\nu}. \tag{9}$$

The extension to curved manifolds is straightforward

$$T^{\mu\nu} = (\rho + p/c^2) U^\mu U^\nu - p g^{\mu\nu}, \tag{10}$$

just as the formula for the **local conservation of energy**

$$\partial_\mu T^{\mu\nu} = 0 \quad \longrightarrow \quad \nabla_\mu T^{\mu\nu} = 0. \tag{11}$$

2 Stress-energy tensor for electromagnetic energy

Here, I briefly summarize some basic results of EM theory. First, recall that the **electromagnetic field tensor** is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{12}$$

which in matrix representation, recovering the electric and magnetic fields, is

$$[F_{\mu\nu}] = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{pmatrix} \tag{13}$$

The Lagrangian density of EM can be written using the EM field tensor and the **electric current density** 4-vector $J^\mu = (c\rho, \mathbf{J})$

$$\mathcal{L} = -\frac{1}{4\mu_0} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu \tag{14}$$

Using the EM Lagrangian density in free space (no charges), we can derive the Hamiltonian density and therefore the EM stress-energy tensor

$$[T^{\mu\nu}] = \begin{pmatrix} \varepsilon_0 E^2/2 + B^2/2\mu_0 & S_1/c & S_2/c & S_3/c \\ S_1/c & -\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ S_2/c & -\sigma_{21} & -\sigma_{22} & -\sigma_{23} \\ S_3/c & -\sigma_{31} & -\sigma_{32} & -\sigma_{33} \end{pmatrix} \tag{15}$$

where

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} \tag{16}$$

is the Poynting vector, and

$$\sigma_{ij} = \varepsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right) \delta_{ij} \tag{17}$$

is the Maxwell stress tensor (related to the Lorentz force, important in the evaluation of the mechanical effects of EM forces).

3 Symmetry of the stress-energy tensor

The stress-energy tensor is a symmetric tensor both in the dust (and perfect fluid) formulation, and in the EM case. This is a general property, associated with the meaning of the different components of the tensor. It can be proved assuming that the tensor is not symmetric: it turns out that this produces absurd physical results and the non-symmetric case is discarded as non-physical.

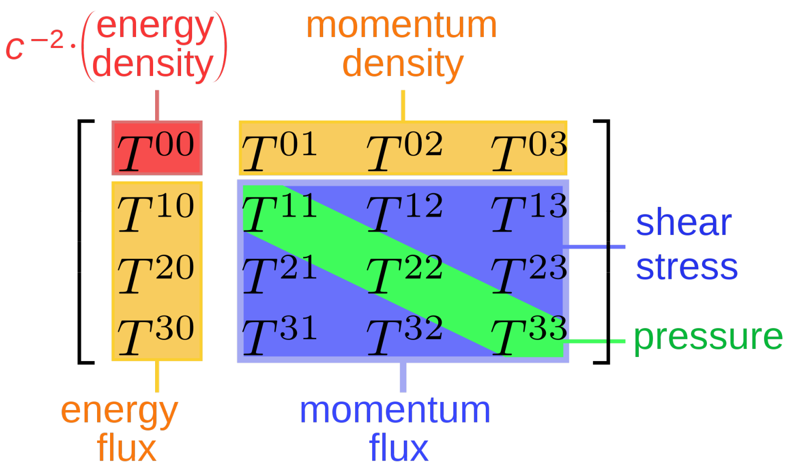

Figure 1: Simple mnemonic diagram to remember the meaning of the components of the stress-energy tensor in contravariant form (from Wikipedia).