Fourier e la soluzione dell'equazione del calore

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Corso di Metodi di Trattamento del Segnale

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Fourier 1807
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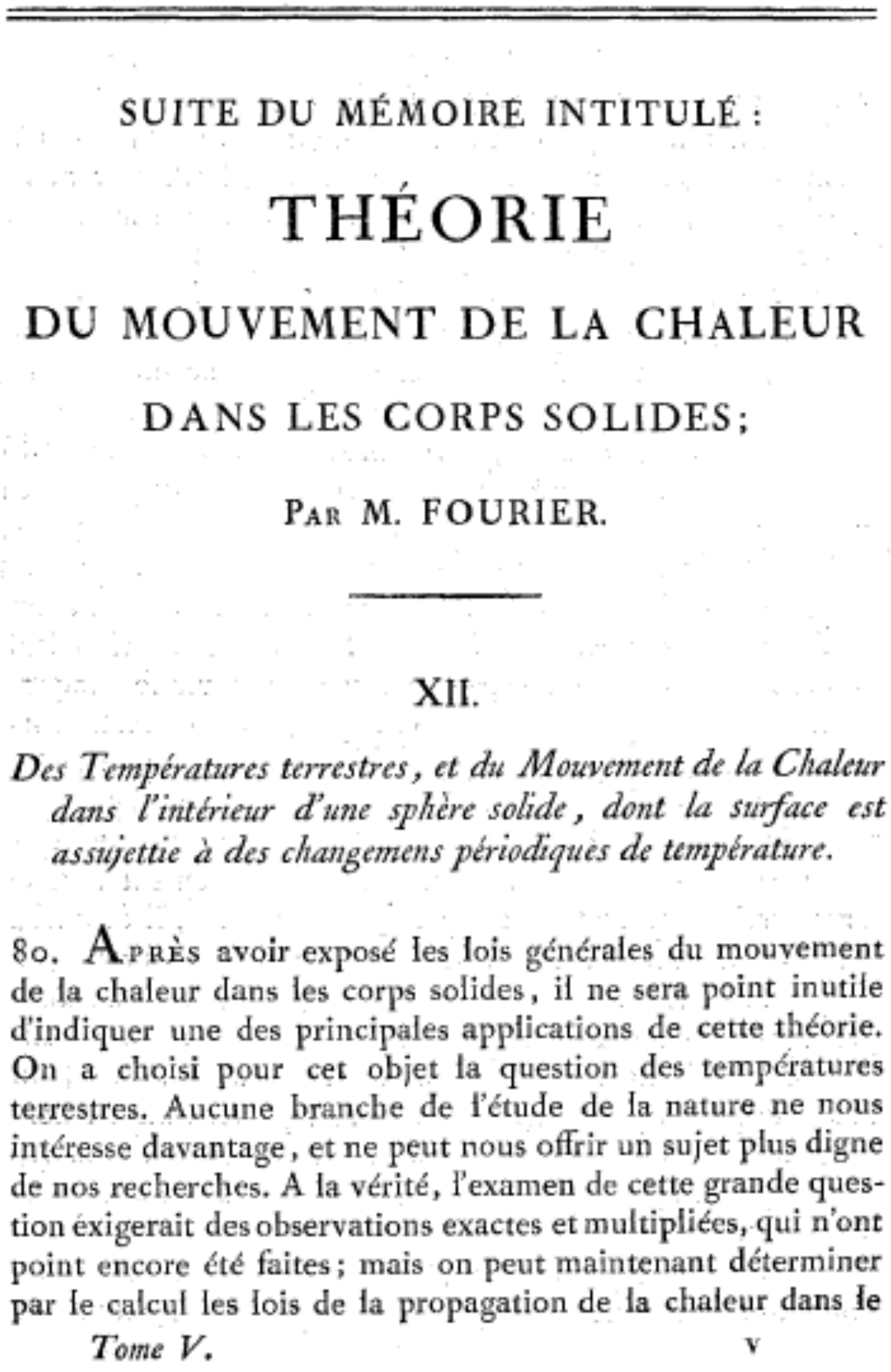


SUITE DU MÉMOIRE INTITULÉ :

THÉORIE

DU MOUVEMENT DE LA CHALEUR DANS LES CORPS SOLIDES;

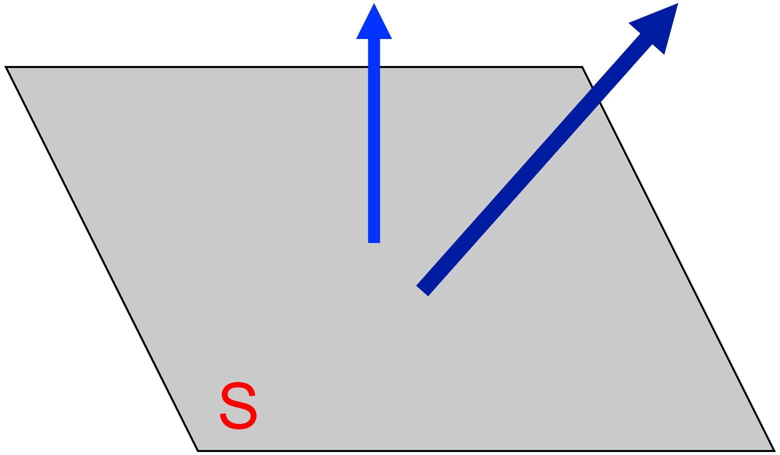
PAR M. FOURIER.

XII.

Des Températures terrestres, et du Mouvement de la Chaleur dans l'intérieur d'une sphère solide, dont la surface est assujettie à des changemens périodiques de température.

80. APRÈS avoir exposé les lois générales du mouvement de la chaleur dans les corps solides, il ne sera point inutile d'indiquer une des principales applications de cette théorie. On a choisi pour cet objet la question des températures terrestres. Aucune branche de l'étude de la nature ne nous intéresse davantage, et ne peut nous offrir un sujet plus digne de nos recherches. A la vérité, l'examen de cette grande question exigerait des observations exactes et multipliées, qui n'ont point encore été faites; mais on peut maintenant déterminer par le calcul les lois de la propagation de la chaleur dans le

Trasporto del calore



Corrente termica = Calore trasportato per unità di tempo

$$= \mathbf{J} \cdot S\mathbf{n}$$

unità di **J**: (Joule/s) / m^2 = W/m^2

Legge di Fick

$$J = -K\frac{dT}{dx}$$

1D

$$\mathbf{J} = -K\nabla T$$

3D

K (conducibilità termica) si misura in W m^{-1} K^{-1}

THERMAL CONDUCTIVITY OF CERAMICS AND OTHER INSULATING MATERIALS (continued)

Material	Dens. g/cm³	t °C	Ther. cond. W/m K
Rock:			
basalt		20	2
chalk		20	0.92
granite	2.8	20	2.2
limestone	2	20	1
sandstone	2.2	20	1.3
slate, ⊥		95	1.4
slate, ∥		95	2.5
Rubber:			
sponge	0.2	20	0.05
92 percent		25	0.16
Sand, dry	1.5	20	0.33
Sawdust	0.2	30	0.06
Shellac		20	0.23
Silica aerogel	0.1	-200 to 20	0.003
Snow	0.25	0	0.16
Steel wool	0.1	55	0.09
Thoria (ThO_2)		100	10
		400	5.8
		1500	2.4
Titanium dioxide		100	6.5
		400	3.8
		1200	3.3
Uranium dioxide		100	9.8
		400	5.5
		1000	3.4
Wood:			
balsa, ⊥	0.11	30	0.04
fir, ⊥	0.54	20	0.14
fir, ∥	0.54	20	0.35
oak		20	0.16
plywood		20	0.11
pine, ⊥	0.45	60	0.11
pine, ∥	0.45	60	0.26
walnut, ⊥	0.65	20	0.14
Wool	0.09	30	0.04
Zinc oxide		200	17
		800	5.3
Zirconia (ZrO_2)		100	2
		400	2
		1500	2.5
Zirconia + silica		200	5.6
		600	4.6
		1500	3.7

THERMAL CONDUCTIVITY OF ALLOYS AS A FUNCTION OF TEMPERATURE

This table lists the thermal conductivity of selected alloys at various temperatures. The indicated compositions refer to weight percent. Since the thermal conductivity is sensitive to exact composition and processing history, especially at low temperatures, these values should be considered approximate.

REFERENCES

1. Powell, R. L., and Childs, G. E., in *American Institute of Physics Handbook, 3rd Edition*, Gray, D. E., Ed., McGraw-Hill, New York, 1972.
2. Ho, C. Y., et al., *J. Phys. Chem. Ref. Data*, 7, 959, 1978.

Thermal conductivity in W/m K

	Alloy	**4 K**	**20 K**	**77 K**	**194 K**	**273 K**	**373 K**	**573 K**	**973 K**
Aluminum:	1100	50	240	270	220	220			
	2024	3.2	17	56	95	130			
	3003	11	58	140	150	160			
	5052	4.8	25	77	120	140			
	5083, 5086	3	17	55	95	120			
	Duralumin	5.5	30	91	140	160	180		
Bismuth:	Rose metal		5.5	8.3	14	16			
	Wood's metal	4	17	23					
Copper:	electrolytic tough pitch	330	1300	550	400	390	380	370	350
	free cutting, leaded	200	800	460	380	380			
	phosphorus, deoxidized	7.5	42	120	190	220			
	brass, leaded	2.3	12	39	70	120			
	bronze, 68% Cu; 32% Zn	2.3	16	48	92	110			
	beryllium	2	17	36	70	90	113	172	
	german silver	0.75	7.5	17	20	23	25	30	40
	silicon bronze A		3.4	11	23	30			
	manganin	0.48	3.2	14	17	22			
	constantan	0.9	8.6	17	19	22			
Ferrous:	commercial pure iron	15	72	106	82	76	66	54	34
	plain carbon steel(AISI 1020)	13	20	58	65	65			
	plain carbon steel(AISI 1095)		8.5	31	41	45			
	3% Ni; 0.7% Cr; 0.6% Mo		6	22		33	35	36	30
	4% Si					20	24	28	26
	stainless steel	0.3	2	8	13	14	16	19	25
	27% Ni; 15% Cr		1.7	55		11	12	16	21

Equazione di continuità

$$-\frac{\partial}{\partial t}\int_V U\,dV = \int_S \mathbf{J}\cdot d\mathbf{S} = \int_V \nabla\cdot\mathbf{J}\,dV$$

$$-\frac{\partial U}{\partial t} = \nabla\cdot\mathbf{J}$$

Equazione del calore

$$\Delta U = C\Delta T$$

$$-\frac{\partial U}{\partial t} = -C\frac{\partial T}{\partial t} = \nabla\cdot\mathbf{J} = -K\nabla^2 T$$

Equazione del calore

$$\frac{\partial T}{\partial t} = D\nabla^2 T$$

$D = K/C$ Costante di diffusione

HEAT CAPACITY OF SELECTED SOLIDS

This table gives the molar heat capacity at constant pressure of representative metals, semiconductors, and other crystalline solids as a function of temperature in the range 200 to 600 K.

REFERENCES

1. Chase, M. W., et al., *JANAF Thermochemical Tables, 3rd ed., J. Phys. Chem. Ref. Data,* 14, Suppl. 1, 1985.
2. Garvin, D., Parker, V. B., and White, H. J., *CODATA Thermodynamic Tables,* Hemisphere Press, New York, 1987.
3. DIPPR Database of Pure Compound Properties, Design Institute for Physical Properties Data, American Institute of Chemical Engineers, New York, 1987.

	C_p in J/mol K						
Name	200 K	250 K	300 K	350 K	400 K	500 K	600 K
Aluminum	21.33	23.08	24.25	25.11	25.78	26.84	27.89
Aluminum oxide	51.12	67.05	79.45	88.91	96.14	106.17	112.55
Anthracene	138.6	173.9	210.7	248.8	288.4		
Benzoic acid	102.7	123.5	147.4	172.0			
Beryllium	9.98	13.58	16.46	18.53	19.95	21.94	23.34
Biphenyl	131.0	162.5	197.2				
Boron	5.99	8.82	11.40	13.65	15.69	18.72	20.78
Calcium	24.54	25.41	25.94	26.32	26.87	28.49	30.38
Calcium carbonate	66.50	75.66	83.82	91.51	96.97	104.52	109.86
Calcium oxide	33.64	38.59	42.18	45.07	46.98	49.33	50.72
Cesium chloride	50.13	51.34	52.48	53.58	54.68	56.90	59.10
Chromium	19.86	22.30	23.47	24.39	25.23	26.63	27.72
Cobalt	22.23	23.98	24.83	25.68	26.53	28.20	29.66
Copper	22.63	23.77	24.48	24.95	25.33	25.91	26.48
Copper oxide	34.80		42.41	44.95	46.78	49.19	50.83
Copper sulfate	77.01	89.25	99.25	107.65	114.93	127.19	136.31
Germanium			23.25	23.85	24.31	24.96	25.45
Gold			25.41	25.37	25.51	26.06	26.65
Graphite	5.01	6.82	8.58	10.24	11.81	14.62	16.84
Hexachlorobenzene	162.7	183.6	202.4				

Calcolo della costante di diffusione del calore per la roccia calcarea

Densità del calcare $\approx 2.7 \cdot 10^{3}$ kg m^{-3}

Peso molecolare ($CaCO_3$) $\approx$ 100 Da

Capacità termica $C \approx 80$ J $mole^{-1}$ K^{-1} $= 0.8$ kJ kg^{-1} K^{-1}

$= 2.16 \cdot 10^{6}$ J m^{-3} K^{-1}

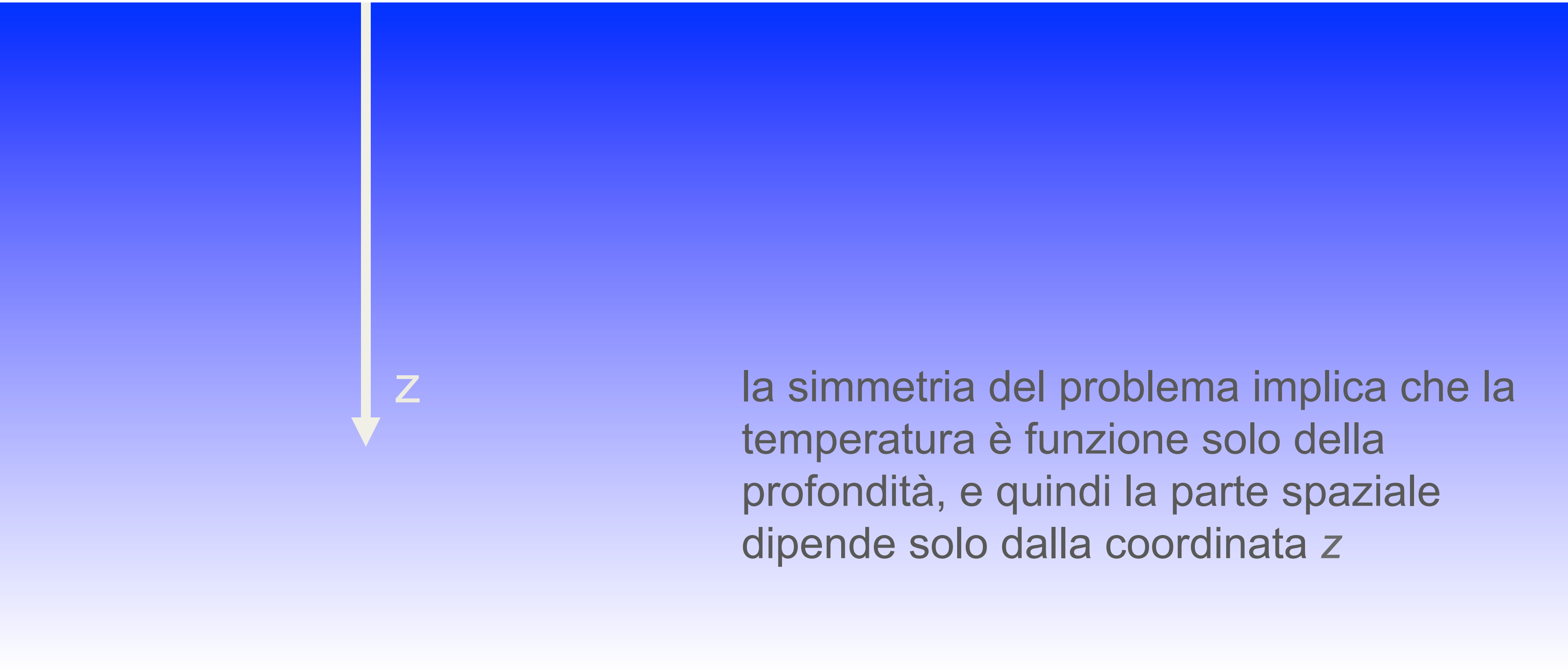
Conducibilità termica $K \approx 1$ W m^{-1} K^{-1}

quindi:

$$D = K/C \approx 0.46 \cdot 10^{-6}\ \mathrm{m^2/s}$$

Equazione del calore in un semispazio

su questo bordo la temperatura viene fissata dall’esterno



la simmetria del problema implica che la temperatura è funzione solo della profondità, e quindi la parte spaziale dipende solo dalla coordinata *z*

Soluzione con condizione al contorno specificata

$$\Delta T(0,t) = \Delta T_0 \cos(\omega t)$$

Condizione al contorno

$$\Delta T(z,t) = \tau(z)\cos\left[\omega t + \varphi(z)\right]$$

Soluzione di prova

$$\Delta \hat{T}(z,t) = \hat{\tau}(z)e^{i\omega t}$$

Estensione complessa

Soluzione di prova nell'equazione del calore

$$\frac{\partial T}{\partial t} = D\nabla^2 T$$

Sol. di prova: $\Delta\hat{T}(z,t) = \hat{\tau}(z)e^{i\omega t}$

$$i\omega\hat{\tau}(z)e^{i\omega t} = D\frac{d^2\hat{\tau}}{dz^2}e^{i\omega t}$$

$$i\omega\hat{\tau}(z) = D\frac{d^2\hat{\tau}}{dz^2}$$

Soluzione ...

$$\hat{\tau}(z) = \left(\frac{D}{i\omega}\right)\frac{d^2\hat{\tau}}{dz^2}$$

$$\hat{\tau}(z) = \hat{A}e^{+ik_0 z} + \hat{B}e^{-ik_0 z}$$

$$k_0 = \sqrt{-\frac{i\omega}{D}}$$

Frequenza spaziale complessa

Frequenza spaziale complessa

$$k_0 = \sqrt{-\frac{i\omega}{D}} = \sqrt{\frac{\omega}{D}e^{-i\pi/2}} = \sqrt{\frac{\omega}{D}}e^{-i\pi/4}$$

$$= \sqrt{\frac{\omega}{D}}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$= \sqrt{\frac{\omega}{2D}}(1-i)$$

Soluzione della parte spaziale

$$\hat{\tau}(z) = \hat{A}e^{+\sqrt{\frac{\omega}{2D}}(1-i)z} + \hat{B}e^{-\sqrt{\frac{\omega}{2D}}(1-i)z} \Rightarrow \hat{A} = 0$$

$$\hat{\tau}(0) = \hat{B} = \Delta T_0$$

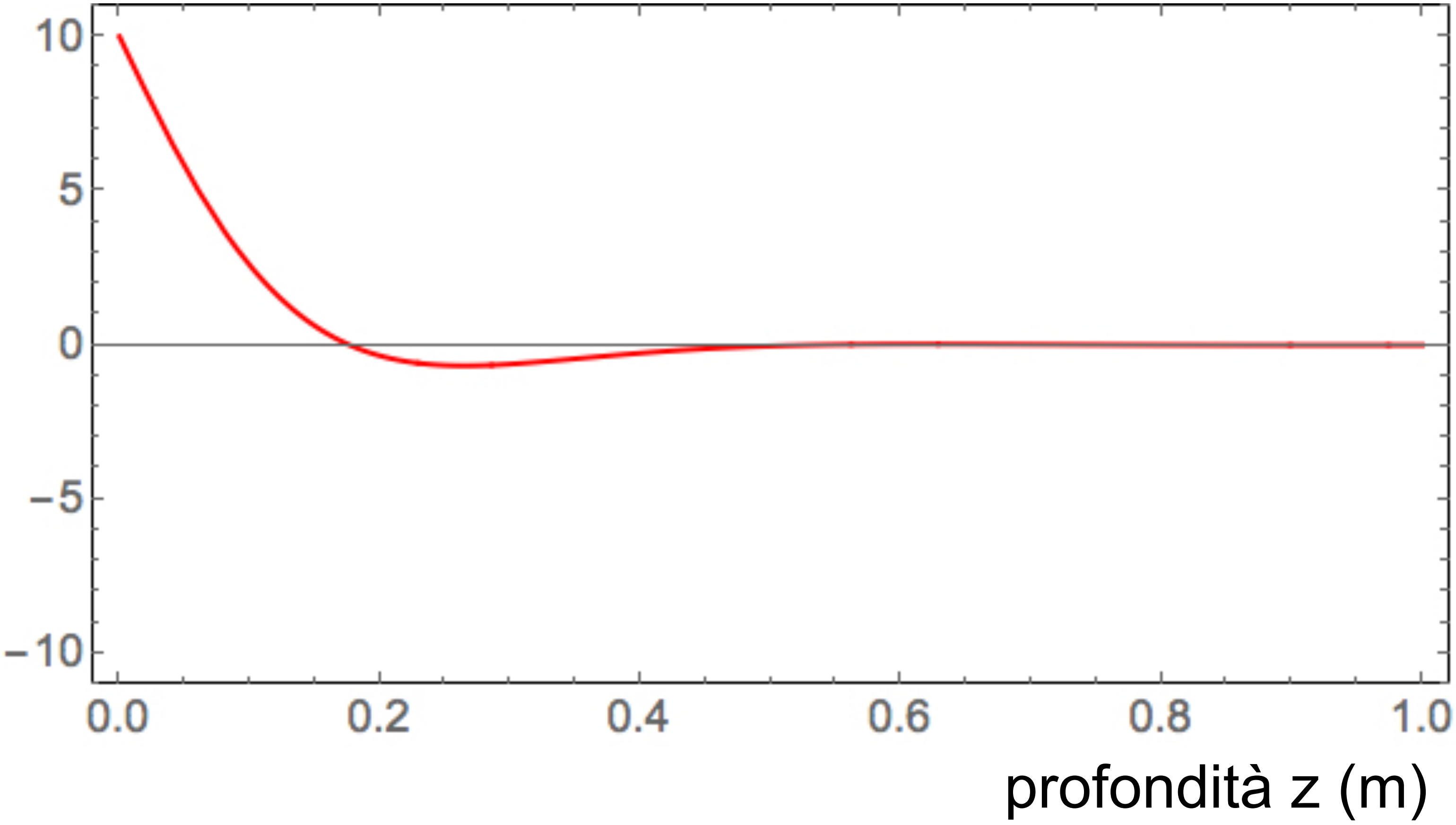
$$\hat{\tau}(z) = \Delta T_0 e^{\sqrt{\frac{\omega}{2D}}(i-1)z}$$

Soluzione completa

(solo parte reale della soluzione spaziale)

$$\Delta T(z,t) = \Delta T_0 e^{-\sqrt{\frac{\omega}{2D}}z} \cos\left[\sqrt{\frac{\omega}{2D}}z\right]\cos(\omega t)$$

ΔT vs. z (m)



La linearità dell'equazione differenziale di partenza implica che quando la condizione al contorno contiene due componenti di Fourier, allora anche la soluzione contiene due componenti di Fourier indipendenti (principio di sovrapposizione)

$$\begin{aligned}\Delta T(z,t) = {} & \Delta T_1 e^{-\sqrt{\frac{\omega_1}{2D}}z} \cos\left[\sqrt{\frac{\omega_1}{2D}}z\right]\cos(\omega_1 t) \\ & + \Delta T_2 e^{-\sqrt{\frac{\omega_2}{2D}}z} \cos\left[\sqrt{\frac{\omega_2}{2D}}z\right]\cos(\omega_2 t)\end{aligned}$$

Temperatura a varie profondità

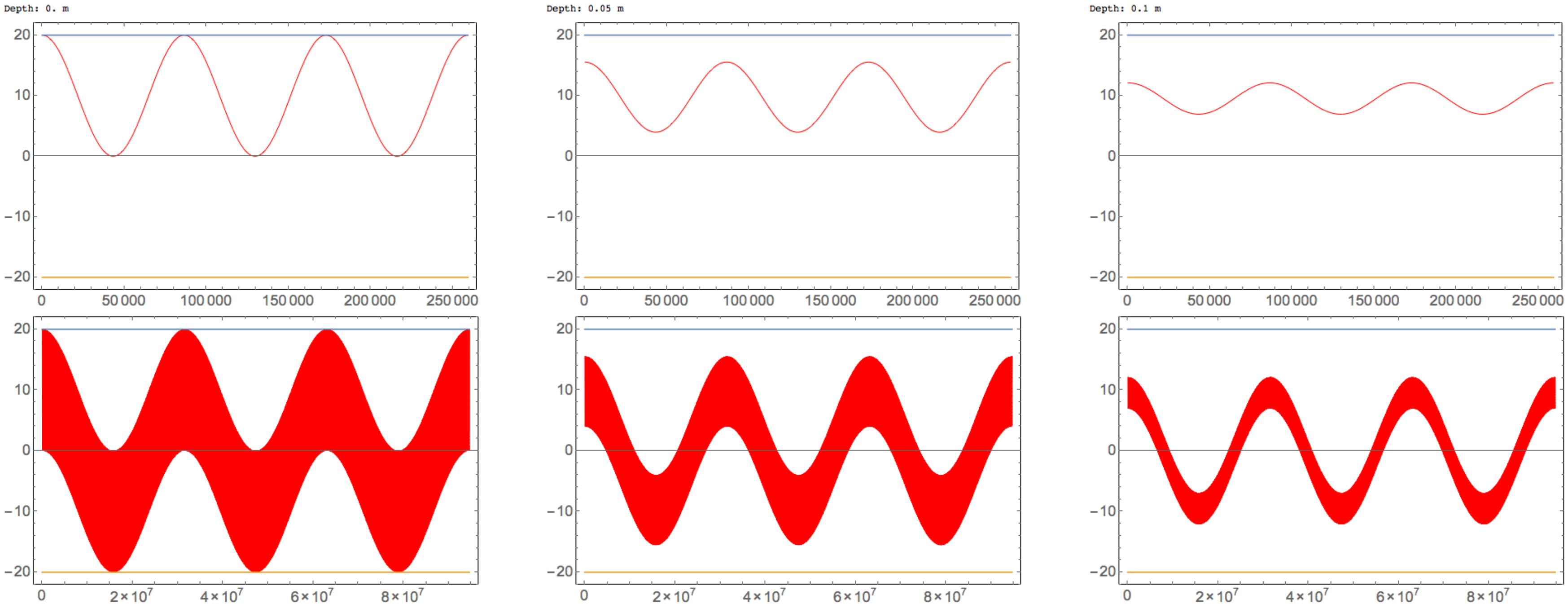
0 cm

5 cm

10 cm

3d

3y



Temperatura a varie profondità

0 cm

25 cm

50 cm

3d

3y

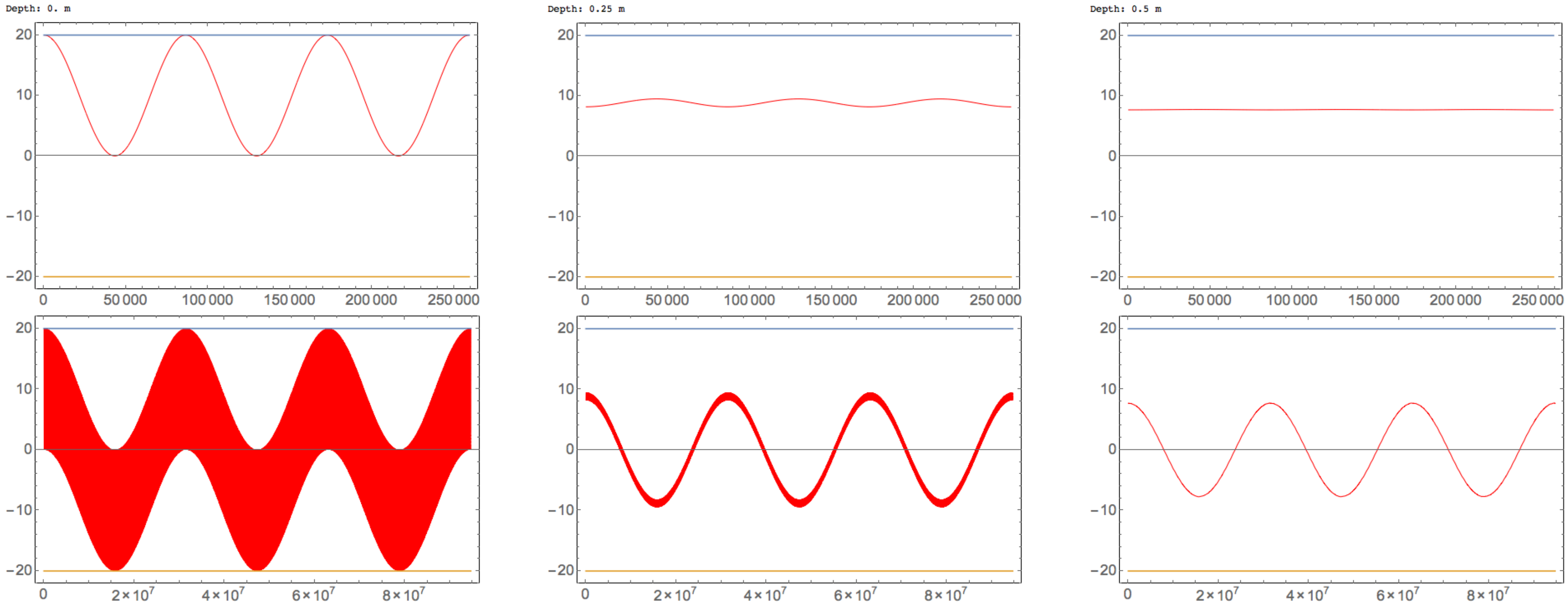
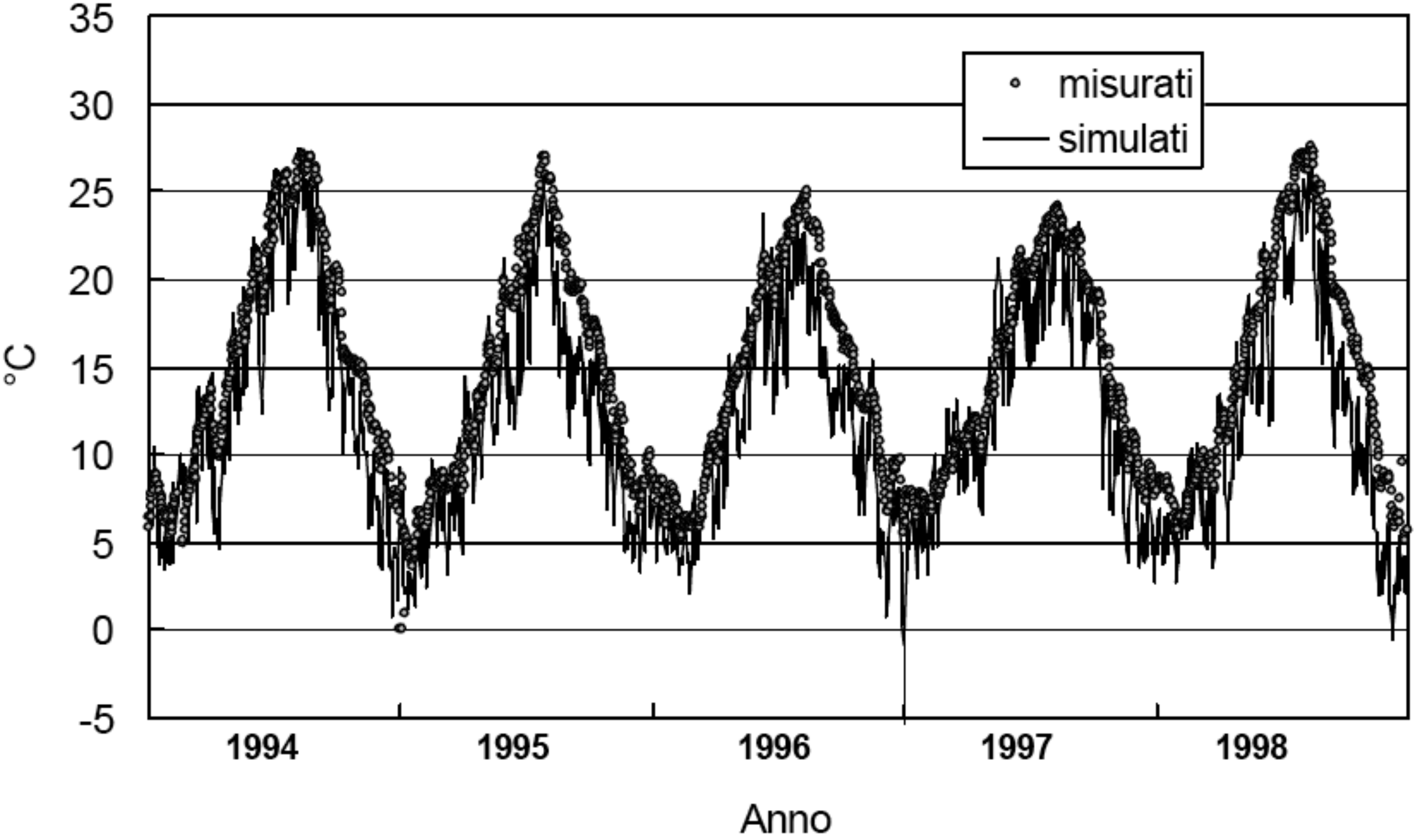


Figura 4 - Stazione di Cesa: confronto tra valori di temperatura del suolo, a 0,4 m di profondità, misurati e simulati con Epic.



E. A. C. Costantini et al.: “*Regime termico del suolo in alcuni campi sperimentali del nord, centro e sud Italia*”, rapporto interno del Centro Nazionale Cartografia Pedologica, 1999

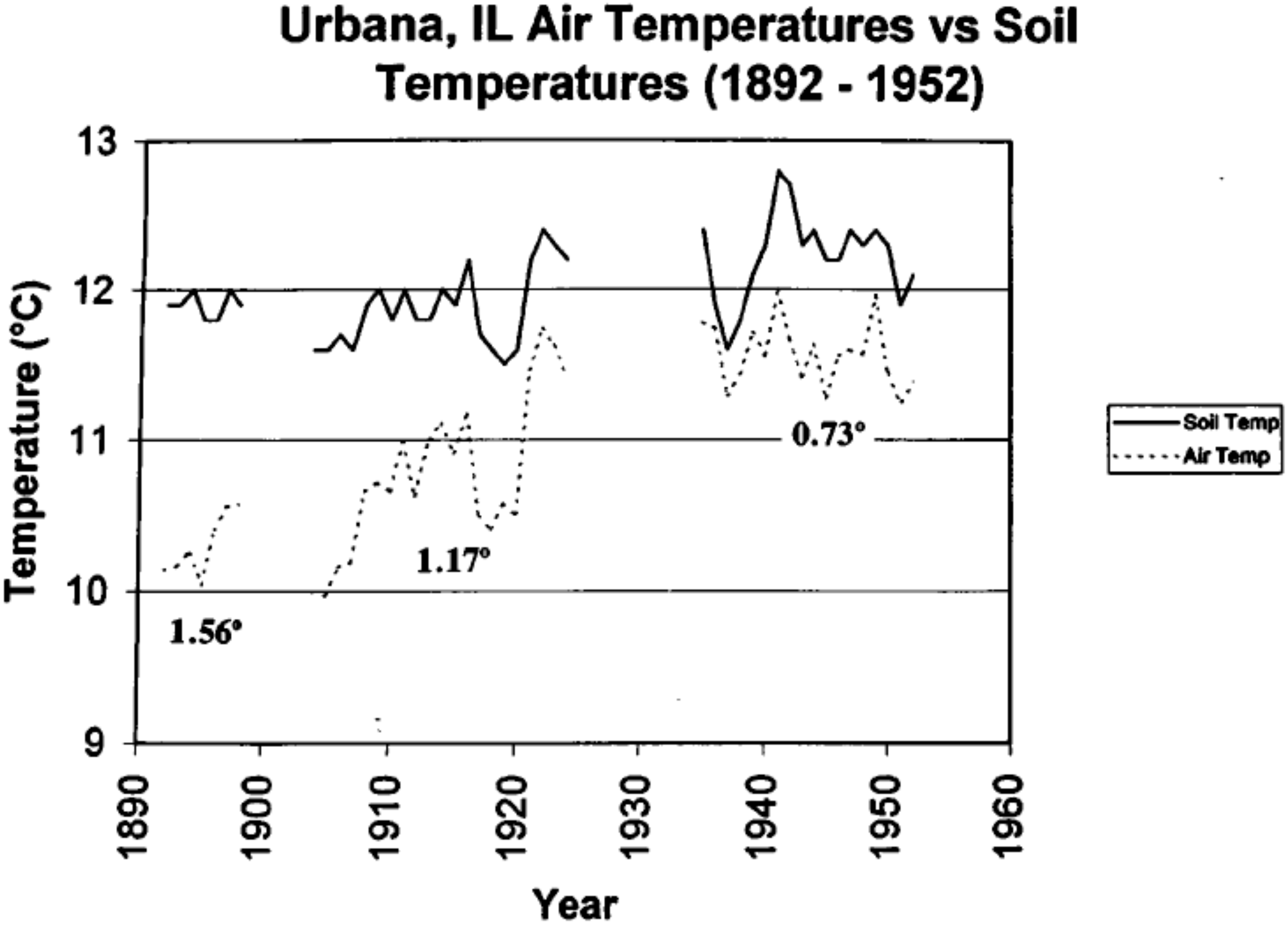


Figure 1. Soil (0.91 m depth) and surface air temperatures at Urbana-Champaign, Illinois, from 1898 to 1952, expressed as 4-year moving averages plotted at the ending year of the 4-year period. The average difference between the air and soil values for each of the three discrete periods is displayed below each period.

S. A. Changnon, Clim. Change **42** (1999) 531