

Computer Algebra

*Symbols as well as numbers can be manipulated by a computer.
New, general-purpose algorithms can undertake a wide variety
of routine mathematical work and solve intractable problems*

by Richard Pavelle, Michael Rothstein and John Fitch

Of all the tasks to which the computer can be applied none is more daunting than the manipulation of complex mathematical expressions. For numerical calculations the digital computer is now thoroughly established as a device that can greatly ease the human burden of work. It is less generally appreciated that there are computer programs equally well adapted to the manipulation of algebraic expressions. In other words, the computer can work not only with numbers themselves but also with more abstract symbols that represent numerical quantities.

Perhaps one reason the algebraic capabilities of the computer have not been fully exploited is that computer programming itself is much like algebra in character. It might therefore seem there is a natural division of labor: the programmer manipulates algebraic symbols, whereas the computer is confined to arithmetic calculations, or "number crunching." Yet the dichotomy between the programmer and the computer was recognized to be a false one as early as 1844 by Augusta Ada Byron, Countess of Lovelace, the daughter of Lord Byron and possibly the first computer programmer. Lady Lovelace was a benefactor of Charles Babbage's and devised some of the programs for the early mechanical computer that Babbage called the difference engine. She pointed out that the computer could "arrange and combine its numerical quantities exactly as if they were letters or any other general symbols; and in fact it might bring out its results in algebraic notation, were provisions made accordingly." Similarly, the modern digital computer is a general-purpose machine that can carry out any algorithm, or precisely specified procedure. The algorithms of algebra can be executed by a computer as readily as those of arithmetic.

In 1973 one of us (Pavelle) undertook an algebraic calculation pertaining to the general theory of relativity; the calculation required three months of work with pencil and paper. The following year a more formidable problem arose in an attempt to define the mathematical

properties that might distinguish general relativity from various other theories of gravitation. Instead of attempting another calculation by hand, Pavelle decided to construct a computer program for manipulating mathematical expressions of the kind that commonly appear in gravitational theories. The program was then written in the computer language of a powerful system of algebraic programs called MACSYMA, then under development at the Massachusetts Institute of Technology. Once Pavelle's program was installed in MACSYMA, the 1973 problem was solved as a test; the computer confirmed the results of the three-month calculation in two minutes. The experience is not unusual among users of computer-algebra systems.

In order to understand the need for automatic systems of algebraic manipulation it must be appreciated that many concepts in science are embodied in mathematical statements where there is little point to numerical evaluation. Consider the simple expression $3\pi^2/\pi$. As any student of algebra knows, the fraction can be reduced by canceling π from both the numerator and the denominator to obtain the simplified form 3π . The numerical value of 3π may be of interest, but it may also be sufficient, and perhaps of greater utility, to leave the expression in the symbolic, nonnumerical form. With a computer programmed to do only arithmetic, the expression $3\pi^2/\pi$ must be evaluated; when the calculation is done with a precision of 10 significant figures, the value obtained is 9.424777958. The number, besides being a rather uninformative string of digits, is not the same as the number obtained from the numerical evaluation (to 10 significant figures) of 3π . The latter number is 9.424777962; the discrepancy in the last two decimal places results from round-off errors introduced by the computer. The equivalence of $3\pi^2/\pi$ and 3π would probably not be recognized by a computer programmed in this way.

The example illustrates three advantages that algebraic programs have over

purely numerical ones. First, it is frequently more economical of computer time to simplify an expression algebraically before evaluating it numerically. Although in this example the saving in computer time is trivial, there are many complicated problems in which the economy that results from algebraic simplification is significant. Second, unlike the numerical approximations generated by a computer, algebraic answers are exact. Approximations necessarily introduce errors; if there are many successive numerical operations, the errors can accumulate and make nonsense of the final result. Only if a careful error analysis is undertaken can the final answer be stated with confidence, and such error analysis is one of the most complex problems faced in many fields.

The third and perhaps the most important advantage is that the goals of scientific investigation are often better served by a result in algebraic form. As Richard W. Hamming of Bell Laboratories has written, "the purpose of computing is insight, not numbers." Insight is sometimes obtained by evaluating a mathematical expression, but in many cases the relations of the quantities are made clearer by algebraic means.

In the study of a chemical process, for example, it is possible to express algebraically the relation between the stability of the process and the relative quantities of the substances present. From such a relation one can predict quite accurately whether a small change in one quantity will cause a violent reaction or a controlled one. Similarly, in the theory of stellar evolution one can examine algebraically how a number of variables determine whether a star will become a neutron star or a black hole, or how the variables interact to predict the existence of a new object.

There is a fourth advantage of the automatic systems we shall call computer-algebra systems, which is brought out by considering some pragmatic aspects of the accumulation of scientific knowledge. A scientific theory is often stated in a concise and quite general mathematical expression that suggests the

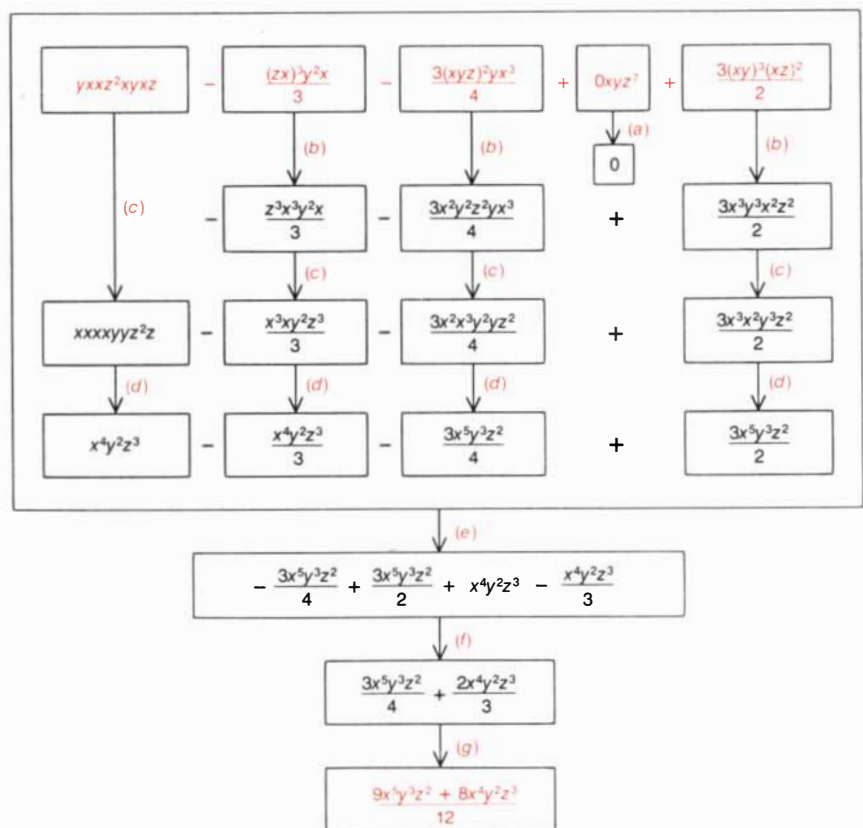
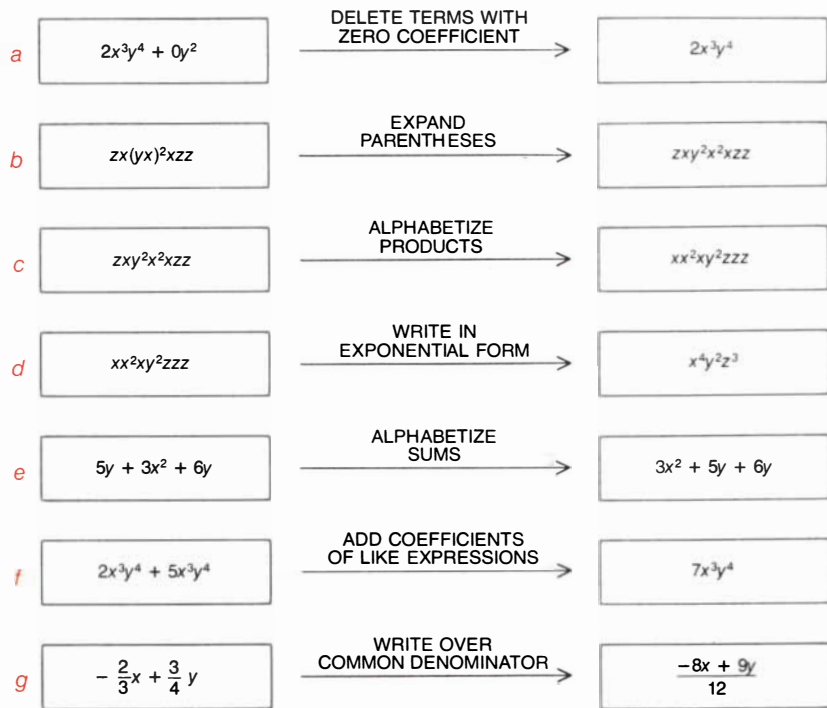
form the theory will take under certain assumptions. For example, a portion of the general theory of relativity, known as the Einstein field equations, can be written down in one line, given certain widely accepted notational conventions. To explore the physical implications of the theory is to explore the mathematical implications of the field equations, but with some exceptions the exploration requires algebraic operations that are intractable if they are done by hand. Even under assumptions that simplify the problem enormously, the algebra is so taxing that few investigators will risk their health, happiness and professional well-being attempting a solution.

The results of such work, even if they are admitted to the main body of scientific knowledge, stand as isolated intellectual outposts, untested by other workers and accepted primarily on the reputation of the investigator rather than on any promise of future critical examination. Such a regrettable state of affairs can prevail in any scientific field at what might be called the frontiers of intractability. For many fields computer algebra has now significantly expanded the frontiers.

In celestial mechanics there is a distinguished tradition of long algebraic calculations. In 1847 the French astronomer Charles Delaunay began to calculate the position of the moon as a function of time. In a sense the calculation is a straightforward application of Newton's theory of universal gravitation. In Newton's theory the orbit of a point mass around a spherical mass of uniform density is an ellipse, but the characteristics of the earth-moon system make the moon's orbit a curve considerably more complex than an ellipse. The plane of the moon's orbit around the earth is inclined at a small angle to the plane of the earth's orbit around the sun, and the angle varies under the perturbing influence of the sun's gravitational field. The sun also causes the lunar perigee (the point in the moon's orbit where it is closest to the earth) to precess slowly with respect to the stars.

Because of such complications the position of the moon has often been determined not from some function of time, as Delaunay set out to do, but by numerical extrapolation from a previous lunar position. Over short periods the errors accumulated in the extrapolation are small, and a correction can be made by again determining the position of the moon by observation. Calculating fast enough to keep up with the progress of the moon is fairly difficult even for the fastest computers, and the required observations are both time-consuming and costly. Delaunay's calculation avoided these difficulties, but it had one serious shortcoming: it demanded 20 years of his life to carry out and check.

The results of Delaunay's work were



COMPUTER ALGORITHM for simplifying an algebraic expression by combining like terms closely resembles the standard manual method of simplification. For the computer each step of the procedure must be precisely specified. In the upper diagram the effect of each step on a simple algebraic expression is illustrated. In the lower diagram the application of the algorithm to the more complicated expression shown in color is illustrated by a flow chart. The colored letters that label the arrows in the flow chart correspond to the elementary operations in the upper diagram. The simplified form of the expression generated by the algorithm is given in color at the bottom. Simplification is one of the tasks that must be done by any set of computer programs intended to solve mathematical problems expressed in symbols rather than numerical quantities. Although the simplification algorithm usually yields a useful and manageable algebraic expression, no single algorithm can reduce every expression to its simplest form.

```

(C1) F(J):=FACTOR(NUSUM(K^J,K,O,N));
(D1) F(J):=FACTOR(NUSUM(K^J,K,O,N))
(C2) FOR J:1 THRU 2 DO DISPLAY (SUM(K^J,K,O,N)=F(J));
(D2)
      N
      ****
      \
      >      N(N+1)
      /      2
      ****
      K = 0

      N
      ****
      \
      >      N(N+1)(2N+1)
      /      6
      ****
      K = 0

(C1) (X+7)^10;
(D1) (X+7)^10
(C2) EXPAND(D1);
(D2)
      X^10 + 70 X^9 + 2205 X^8 + 41160 X^7 + 504210 X^6 + 4235364 X^5
      + 24706290 X^4 + 98825160 X^3 + 259416045 X^2 + 403536070 X + 282475249

(C1) X^3+B*X^2+A^2*X^2-9*A*X^2+A^2*B*X-2*A*B*X-9*A^3*X+14*A^2*X-2*A^3*B+14*A^4=0;
(D1) X^3 + B X^2 + A^2 X^2 - 9 A X^2 + A^2 B X - 2 A B X
      - 9 A^3 X + 14 A^2 X - 2 A^3 B + 14 A^4 = 0
(C2) SOLVE(D1,X);
(D2) [X = 7 A - B, X = - A^2, X = 2 A]

(D1)
      1125899906842624 Y^40 + 9007199254740992 Y^39 + 43523068273885184 Y^38
      + 96686654500110336 Y^37 + 71892942171668480 Y^36 - 203545990580404224 Y^35
      - 3231739551940608 Y^34 + 2967153761027358720 Y^33 + 3933037175129505792 Y^32
      - 10392801849559220224 Y^31 - 26535490294760079360 Y^30 + 60970867840763559936 Y^29
      + 124971270109665427456 Y^28 - 294061554052926275584 Y^27 - 377892103185282105344 Y^26
      + 1043007763435371888640 Y^25 + 1247934126027765186560 Y^24 - 4589204662544561602560 Y^23
      - 534479284997751570432 Y^22 + 13715332431765689073664 Y^21 - 13302644475428239835136 Y^20
      - 11542363148164431609856 Y^19 + 30184905793343524306944 Y^18 - 11553929011986625003520 Y^17
      - 20761961484788844621824 Y^16 + 23716893785532021784576 Y^15 - 1470688776981600935936 Y^14
      - 11829726550983847370752 Y^13 + 7047410412446462752768 Y^12 + 590648324272550133760 Y^11
      - 2234695017099712874496 Y^10 + 836560570027035444224 Y^9 + 38308661311225029504 Y^8
      - 113171442907550345472 Y^7 + 29418059499453088640 Y^6 + 66680780706292864 Y^5
      - 1434405693611092512 Y^4 + 283861281845610384 Y^3 - 20855579198349088 Y^2
      + 370175331659648 Y - 1886186974063

(C2) FACTOR(D1);
Time = 130855 msec.

(D2)
      (4096 Y^10 + 8192 Y^9 - 3008 Y^8 - 30848 Y^7 + 21056 Y^6 + 146496 Y^5
      - 221360 Y^4 + 1232 Y^3 + 144464 Y^2 - 78488 Y + 11993)
      *(4096 Y^10 + 8192 Y^9 + 1600 Y^8 - 20608 Y^7 + 20032 Y^6 + 87360 Y^5
      - 105904 Y^4 + 18544 Y^3 + 11888 Y^2 - 3416 Y + 41)
      *(8192 Y^10 + 12288 Y^9 + 66560 Y^8 - 22528 Y^7 - 138240 Y^6 + 572928 Y^5
      - 90496 Y^4 - 356032 Y^3 + 113032 Y^2 + 23420 Y - 8179)
      *(8192 Y^10 + 20480 Y^9 + 58368 Y^8 - 161792 Y^7 + 198656 Y^6 + 199680 Y^5
      - 414848 Y^4 - 4160 Y^3 + 171816 Y^2 - 48556 Y + 469)

```

COMPUTER-ALGEBRA SYSTEM called **MACSYMA**, developed at the Massachusetts Institute of Technology, is a general-purpose set of programs that can solve problems in algebra far beyond the abilities of most mathematicians. The system communicates with the user in a conversational style. After logging in to the system the user types the algebraic expression to be manipulated on the *C* line, or command line, in a computer language that resembles ALGOL or FORTRAN. The system responds on the *D* line with a rewritten version of the expression that is closer to the way it would appear on a printed page. The user then types a second command, and the computer carries out the indicated operation. In the first two examples the computer finds an expression for the sum of the first *n* terms of a series, where each term has the form of the expression next to the letter Σ . (Σk^2 is a way of writing $1^2 + 2^2 + 3^2 + \dots + n^2$.) In the next two examples the computer expands a binomial expression and solves a nonnumerical cubic equation exactly. The upward-pointing arrows indicate exponentiation and the asterisks indicate multiplication. The final example illustrates some of the computational power of the system. The maximum area of a hexagon inscribed in a circle with a diameter of 1 is one of the roots of the enormous 40th-degree polynomial shown. Manual methods of finding roots are unable to cope with a polynomial of this size; it is easier to factor the polynomial and find the roots of the factors. The computer-algebra system gives the most complete factorization whose terms have integral coefficients. A computer can solve the problem in about two minutes.

published in two volumes in 1867, and they remained unchallenged for more than 100 years. During World War II there was interest in expressing the coordinates of fixed stars as a function of time in order to help in determining the positions of German U-boats, but there was no further work on the orbit of the moon or of other bodies in the solar system. With the advent of artificial earth satellites, however, Delaunay's method for determining orbits became economically attractive.

In order to devise algebraic expressions for satellite orbits, André Deprit, Jacques Henrard and Arnold Rom, who were then at the Boeing Scientific Research Laboratories in Seattle, began to investigate algorithms for doing the Delaunay calculation with a computer. In 1970 they made the calculation in about 20 hours of computer time. Remarkably, they found only three errors in Delaunay's work, all in terms of small value; moreover, two of the errors were mere consequences of the third one. Computer-algebra systems have since been developed that extend Delaunay's method by taking into account the effects of atmospheric drag and the non-spherical shape of the earth on a satellite in a low-altitude orbit. The algorithms are now sometimes employed in satellite tracking.

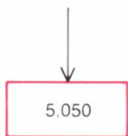
Considering the human life span, a reduction in the time needed for an operation from 20 years to 20 hours is a qualitative gain. Problems as complex as Delaunay's problem can become the focus of research for an investigator who has far less patience than Delaunay had. Moreover, the investigator can spend a much greater proportion of his time weighing the significance of his algebraic results. He can experiment with elaborate calculations without betting his career on the correctness and the utility of the answer. Perhaps the most telling indication that computer algebra has shifted the frontier of intractability in celestial mechanics is that Delaunay's huge calculation has been undertaken as a test of new computer-algebra systems.

How can a computer be programmed to carry out algebraic manipulations? Many of the algorithms employed in computer algebra began as the basic procedures devised by earlier generations of mathematicians that are now taught to algebra students in high school and college. For example, the methods for solving equations in which a single unknown quantity is raised to the first or the second power have been known since the time of Hammurabi, about 1750 B.C. Unlike the student who may develop an intuitive but vague sense of how to proceed through such a solution, however, the computer must follow a rigorous procedure that specifies at every point what the computer is to do.

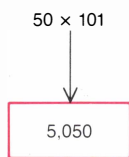
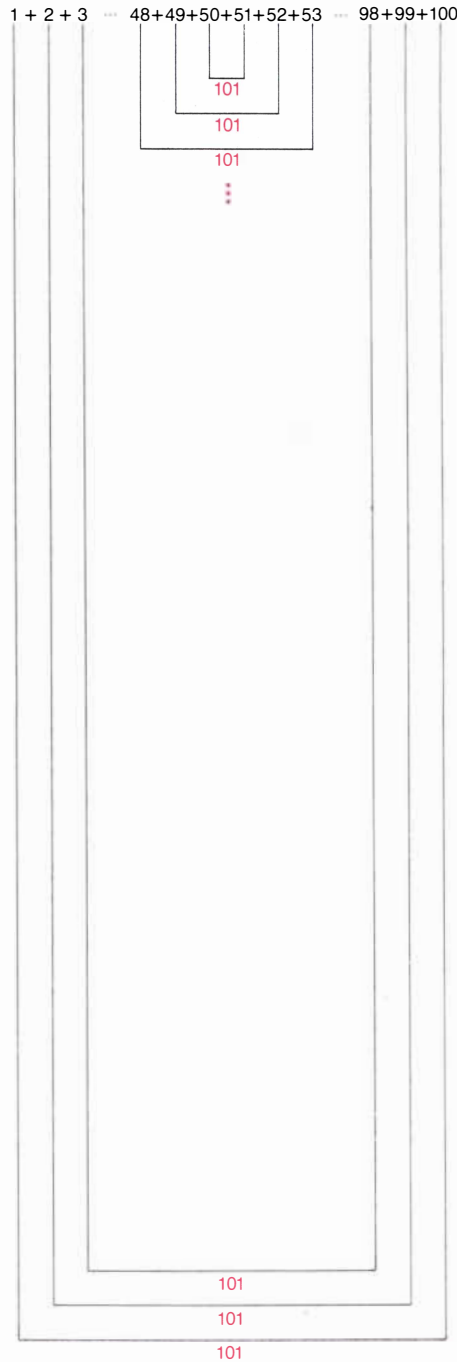
Consider how a student with a mod-

NUMBER CRUNCHING

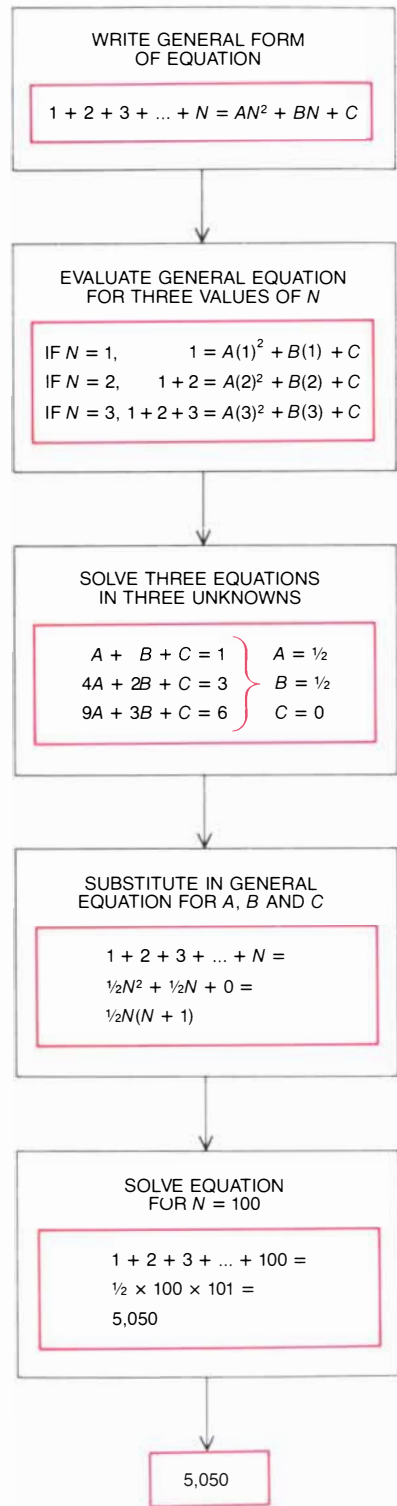
1 + 2 = 3	1,275 + 51 = 1,326
3 + 3 = 6	1,326 + 52 = 1,378
6 + 4 = 10	1,378 + 53 = 1,431
10 + 5 = 15	1,431 + 54 = 1,485
15 + 6 = 21	1,485 + 55 = 1,540
21 + 7 = 28	1,540 + 56 = 1,596
28 + 8 = 36	1,596 + 57 = 1,653
36 + 9 = 45	1,653 + 58 = 1,711
45 + 10 = 55	1,711 + 59 = 1,770
55 + 11 = 66	1,770 + 60 = 1,830
66 + 12 = 78	1,830 + 61 = 1,891
78 + 13 = 91	1,891 + 62 = 1,953
91 + 14 = 105	1,953 + 63 = 2,016
105 + 15 = 120	2,016 + 64 = 2,080
120 + 16 = 136	2,080 + 65 = 2,145
136 + 17 = 153	2,145 + 66 = 2,211
153 + 18 = 171	2,211 + 67 = 2,278
171 + 19 = 190	2,278 + 68 = 2,346
190 + 20 = 210	2,346 + 69 = 2,415
210 + 21 = 231	2,415 + 70 = 2,485
231 + 22 = 253	2,485 + 71 = 2,556
253 + 23 = 276	2,556 + 72 = 2,628
276 + 24 = 300	2,628 + 73 = 2,701
300 + 25 = 325	2,701 + 74 = 2,775
325 + 26 = 351	2,775 + 75 = 2,850
351 + 27 = 378	2,850 + 76 = 2,926
378 + 28 = 406	2,926 + 77 = 3,003
406 + 29 = 435	3,003 + 78 = 3,081
435 + 30 = 465	3,081 + 79 = 3,160
465 + 31 = 496	3,160 + 80 = 3,240
496 + 32 = 528	3,240 + 81 = 3,321
528 + 33 = 561	3,321 + 82 = 3,403
561 + 34 = 595	3,403 + 83 = 3,486
595 + 35 = 630	3,486 + 84 = 3,570
630 + 36 = 666	3,570 + 85 = 3,655
666 + 37 = 703	3,655 + 86 = 3,741
703 + 38 = 741	3,741 + 87 = 3,828
741 + 39 = 780	3,828 + 88 = 3,916
780 + 40 = 820	3,916 + 89 = 4,005
820 + 41 = 861	4,005 + 90 = 4,095
861 + 42 = 903	4,095 + 91 = 4,186
903 + 43 = 946	4,186 + 92 = 4,278
946 + 44 = 990	4,278 + 93 = 4,371
990 + 45 = 1,035	4,371 + 94 = 4,465
1,035 + 46 = 1,081	4,465 + 95 = 4,560
1,081 + 47 = 1,128	4,560 + 96 = 4,656
1,128 + 48 = 1,176	4,656 + 97 = 4,753
1,176 + 49 = 1,225	4,753 + 98 = 4,851
1,225 + 50 = 1,275	4,851 + 99 = 4,950
	4,950 + 100 = 5,050



INSIGHT



ALGEBRAIC ALGORITHM



APPLICATION OF COMPUTER ALGEBRA to a numerical calculation can save computer-processing time by simplifying an algebraic expression before it is evaluated numerically. To find the sum of the first 100 integers a numerical calculation requires 99 separate additions. A simpler algorithm employed by the computer-algebra program solves the problem by taking advantage of a general mathematical result showing that the sum of the first n integers is a quadratic, or second-degree, polynomial function of n . Similarly, the algorithm would find the sum of the squares of the first n integers by constructing a third-degree polynomial function of n . Although the

algorithm is effective, it is not necessarily the fastest way to solve the problem or the one that leads to the clearest understanding of it. According to legend, the German mathematician Carl Friedrich Gauss noticed at the age of seven that the integers from 1 to 100 can be grouped in pairs so that all the pairs add to the same number, namely 101. The sum of the first 100 integers is therefore equal to 101 multiplied by 50, which is the number of pairs. Computer-algebra systems usually cannot recognize such patterns; on the other hand, Gauss's method cannot be applied to finding the sums of squares, cubes or higher powers of integers, as the computer-algebra algorithm can.

A CAR FOR THE LEFT SIDE OF YOUR BRAIN.

The left side of your brain, recent investigations tell us, is the logical side.

It figures out that $1 + 1 = 2$. And, in a few cases, that $E = mc^2$.

On a more mundane level, it chooses the socks you wear, the cereal you eat, and the car you drive. All by means of rigorous Aristotelian logic.

However, and a big however it is, for real satisfaction, you must achieve harmony with the other side of your brain.

The right side, the poetic side, that says, "Yeah, Car X has a reputation for lasting a long time but it's so dull, who'd want to drive it that long anyway?"

The Saab Turbo looked at from all sides.

To the left side of your brain, Saab turbocharging is a technological feat that retains good gas mileage while also increasing performance.

To the right side of your brain, Saab turbocharging is what makes a Saab go like a bat out of hell.

The left side sees the safety in high performance. (Passing on a two-lane highway. Entering a freeway in the midst of high-speed traffic.)

The right side lives only for the thrills.

The left side considers that *Road & Track* magazine just named Saab "The Sports Sedan for the Eighties." By unanimous choice of its editors.

The right side eschews informed endorsements by editors who have spent a lifetime comparing cars. The right side doesn't know much about cars, but knows what it likes.

The left side scans this chart.

Wheelbase	99.1 inches
Length	187.6 inches
Width	66.5 inches
Height	55.9 inches
Fuel-tank capacity	16.6 gallons
EPA City	19 mpg*
EPA Highway	31 mpg*

The right side looks at the picture on the opposite page.

The left side compares a Saab's comfort with that of a Mercedes. Its performance with that of a BMW. Its braking with that of an Audi.

The right side looks at the picture.

The left side looks ahead to the winter when a Saab's front-wheel drive will keep a Saab in front of traffic.

The right side looks at the picture.

The left side also considers the other seasons of the year when a Saab's front-wheel drive gives it the cornering ability of a sports car.

The right side looks again at the picture.

Getting what you need vs. getting what you want.

Needs are boring; desires are what make life worth living.

The left side of your brain is your mother telling you that a Saab is good for you. "Eat your vegetables." (In today's world, you need a car engineered like a Saab.) "Put on your raincoat." (The Saab is economical. Look at the price-value relationship.) "Do your homework." (The passive safety of the construction. The active safety of the handling.)

1982 SAAB PRICE** LIST

900 3-Door	5-Speed	\$10,400
	Automatic	10,750
900 4-Door	5-Speed	\$10,700
	Automatic	11,050
900S 3-Door	5-Speed	\$12,100
	Automatic	12,450
900S 4-Door	5-Speed	\$12,700
	Automatic	13,050
900 Turbo 3-Door	5-Speed	\$15,600
	Automatic	15,950
900 Turbo 4-Door	5-Speed	\$16,260
	Automatic	16,610

All turbo models include a Sony XR70, 4-Speaker Stereo Sound System as standard equipment. The stereo can be, of course, perfectly balanced: left and right.

The right side of your brain guides your foot to the clutch, your hand to the gears, and listens for the "zzzooommm."

Together, they see the 1982 Saab Turbo as the responsible car the times demand you get. And the performance car you've always, deep down, wanted with half your mind.

*Saab 900 Turbo. Remember, use estimated mpg for comparison only. Mileage varies with speed, trip length, and weather. Actual highway mileage will probably be less. **Manufacturer's suggested retail price. Not including taxes, license, freight, dealer charges or options desired by either side of your brain.

**A CAR FOR THE RIGHT SIDE
OF YOUR BRAIN.**



SAAB

The most intelligent car ever built.



She can't wear a Ming vase to La Bohème.



*Born out of fire and ice more than a hundred
million years ago. Every diamond is unique.
But a diamond this large is even more precious.
A gift so rare, it can never be measured.
Until you see the look in her eyes.*

**THE DIAMOND SOLITAIRE.
A RARE GIFT.**

The one and a quarter carat diamond bracelet shown is enlarged for detail.

A diamond is forever. De Beers

erate facility for simple algebra might solve the equation $3x - 1 = 2(x + 5)$. First he would "clear parentheses" by multiplying both terms within the parentheses, x and 5 , by 2 , obtaining $3x - 1 = 2x + 10$. He would then transfer the term $2x$ to the left side of the equation and the term -1 to the right, changing signs as he goes and adding or subtracting terms as necessary. Thus in a single additional step he would obtain the answer $x = 11$.

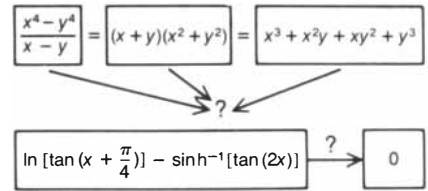
In a computer-algebra program the steps to the solution of the problem have to be specified in somewhat tedious detail, and the shortcuts that might be recognized by a student during work on a particular problem must be spelled out. The first step in solving the equation in our example would probably not differ in the program from the student's method: the program would clear parentheses and obtain the expression $3x - 1 = 2x + 10$. In order to reach the last line, however, the program would have to transfer the terms of the equation one by one to a convenient side of the equal sign. One way to do this would be to program the computer to add an expression called the additive inverse to all the numerical constants appearing on the left side of the equal sign and to all terms that include variables appearing on the right side of the equal sign. Since -1 is the only constant to appear on the left of the equal sign, the computer would add its additive inverse, $+1$, to both sides of the equation, obtaining $3x - 1 + 1 = 2x + 10 + 1$. The program would then rearrange the terms, if necessary, and add or subtract where possible, obtaining $3x = 2x + 11$.

Next, the program would search the expressions on the right of the equal sign for terms including variables and find the additive inverse of the first such term it encountered. In the example the computer would construct the additive inverse of $2x$ (namely $-2x$) and add it to both sides of the equation, thereby obtaining $3x - 2x = 2x + 11 - 2x$. Once again the terms would be rearranged so that like terms could be combined: $3x - 2x = 2x - 2x + 11$. Finally, the program would combine the terms having the same variable names and stop when a single variable appeared on the left side of the equation and a numerical constant appeared on the right: $x = 11$. Such an algorithm is neither the simplest nor the most efficient method one can devise for programming a computer to solve equations that have one unknown quantity, but it is not sufficiently complex either. There are many possibilities, such as the appearance of the expression $2x = 22$ at some step in the solution of the problem, that the algorithm as we have described it is not equipped to deal with.

In developing an algorithm for computer algebra it is not necessary to fol-

low the procedure that is most efficient in manual calculation. Delaunay proved several theorems that he then employed to simplify intermediate calculations. Although the theorems could be incorporated into a program, the effort necessary to express them in algorithmic form encouraged Deprit and his colleagues to search for a method more compatible with mechanized execution. The one they invented requires transformations that would have exceeded the abilities even of Delaunay, but the algorithm is easy to program and can be executed quickly with a computer. The development of new algorithms is one of the most active areas of investigation in computer algebra; it is largely because of such work that computer-algebra systems have been improved significantly in the past few years.

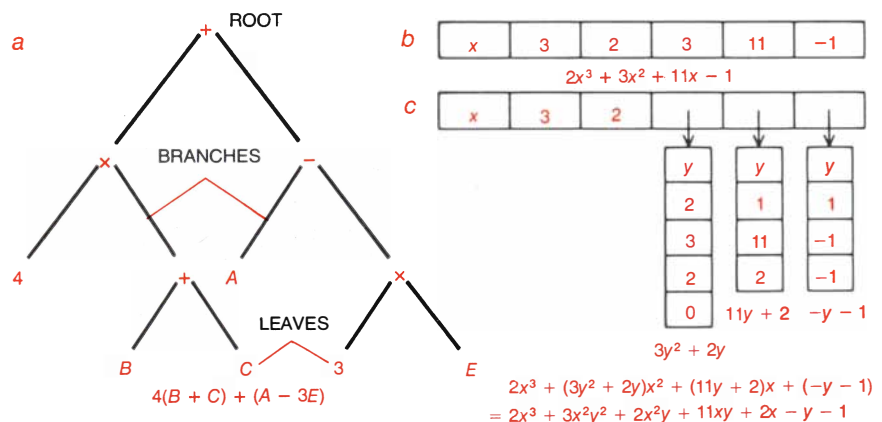
In order to represent an algebraic expression in a computer program, most systems store the minimum information needed to specify the expression uniquely. Current representational schemes employ one of two basic approaches or a combination of the two. In one approach an expression is represented as an inverted treelike structure in which the leaves are the operands. For example, suppose one wished to represent the expression $2(x + 4)$ in a computer. The leaves of the tree would be the terms 2 , x and 4 , although they would appear at different levels. Both the x and the 4 would be connected by upward-moving branches to a plus sign. The symbol 2 , however, would not be linked to the plus sign. Instead branches from the 2 and from the plus sign would



SIMPLIFICATION of an expression presents a problem of choice for a computer-algebra system: given several equivalent forms of an expression, such as those in the upper part of the illustration, which is to be considered the simplest? Many systems now being designed leave the choice partly to the person working with the system. The complex expression in the lower part of the illustration is equal to zero for all values of x for which the tangents are positive real numbers. The expression does not vanish for all values of x , however. If a computer-algebra system were to substitute zero for the expression, the relation of the final answer to the original problem could become quite obscure. The expressions \ln , \tan and \sinh^{-1} are the standard notations respectively for the natural logarithm, the tangent and the inverse hyperbolic sine.

meet at the top, or root, of the tree, which would be labeled with a multiplication sign. The representation makes more efficient the search for subordinate expressions of predetermined form.

In the second scheme "slots" are assigned in some definite order to represent the information carried by an expression. To represent a polynomial in one variable, for instance, one slot is assigned to the name of the variable, the next slot to the degree of the polynomial (the largest power of the variable that appears in the polynomial) and the fol-



REPRESENTING AN ALGEBRAIC FUNCTION in a computer may require several strategies. The inverted tree diagram (a) is a simple and convenient means of representing the sequence of operations that apply to the variables and constants in an expression. A polynomial function of one variable can be represented by a sequence (b) that consists of the name of the variable, the degree of the polynomial (equal to the largest power of the variable that appears in the polynomial) and the numerical values of the coefficients of descending powers of the variable. To represent a polynomial function of two variables (c) a generalization of the polynomial representation for one variable can be employed. Here the horizontal group of slots represents a third-degree polynomial function of the variable x , whereas the vertical groups of slots represent polynomial functions of the variable y . The three polynomials in y are understood to be coefficients, or multipliers, of the terms of the polynomial in x that correspond to the three slots left blank. Hence the coefficient of the term x^2 is $3y^2 + 2y$, the coefficient of the term x is $11y + 2$ and the coefficient of the constant term of the polynomial in x is $-y - 1$.

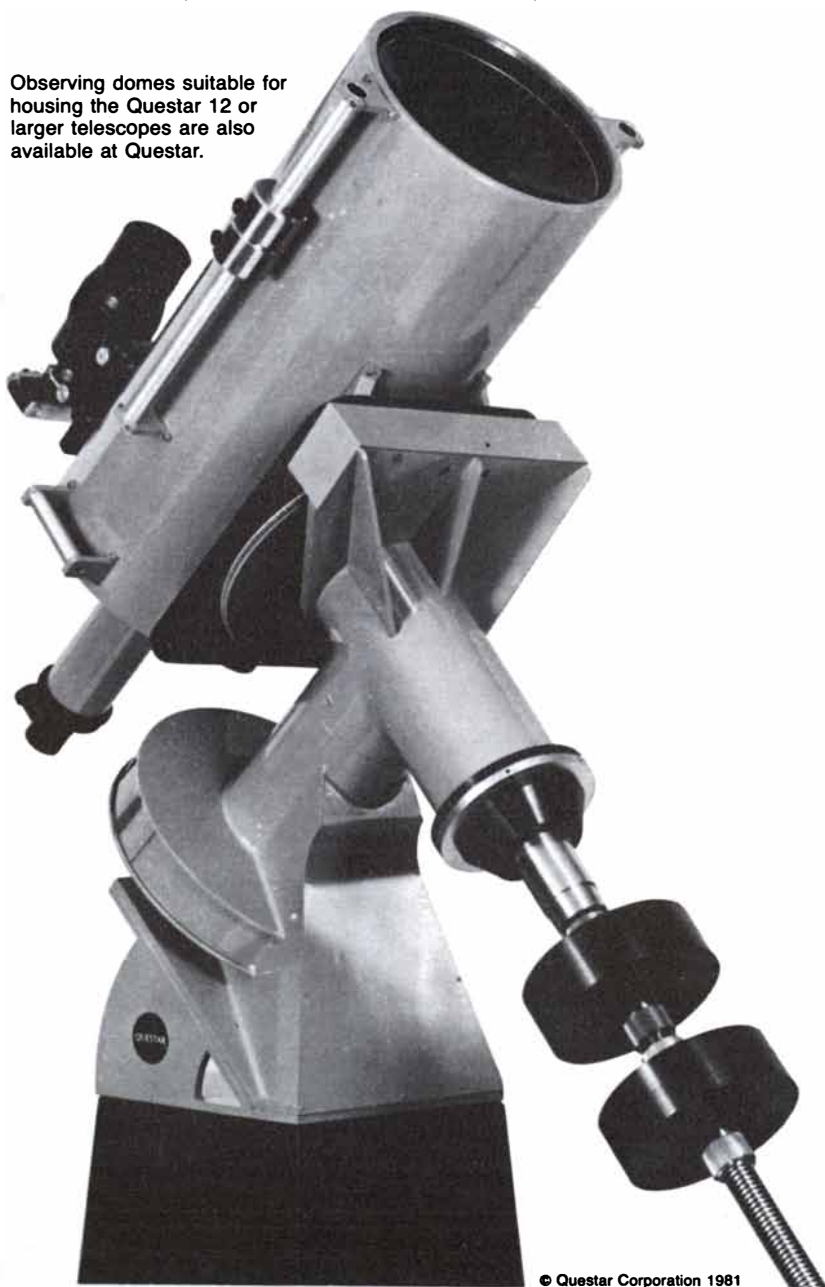
QUESTAR® 12 on the QUESTAR® MOUNT

The Questar 12, latest addition to the Questar family of fine optical instruments, now has its own Questar-designed mount. A German equatorial type, it is notable for its 360° continuous tracking in R.A. with precision tracking to better than 4 arc seconds. The Questar Mount is designed with over-sized components so that it can accommodate any Questar up to 18 inches. The standard mount shown is straightforward in design but can be modified so as to be compatible with more sophisticated tracking devices or other special equipment.

The Questar 12 is a superb instrument for the serious astronomer, for the university astronomy department or the engineer seeking sophisticated tracking and surveillance equipment for which Questar Corporation has a noted reputation.

Questar Corporation Box C, Dept. 20, New Hope, Pa. 18938 (215) 862-5277

Observing domes suitable for housing the Questar 12 or larger telescopes are also available at Questar.

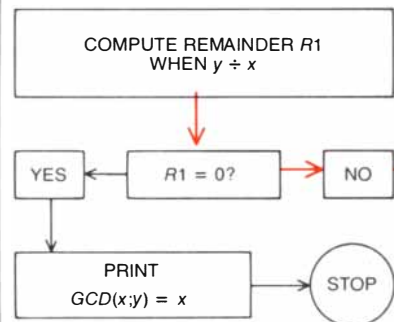


© Questar Corporation 1981

lowing slots to the coefficients of descending powers of the variable. The set of information slots can be made a part of a tree-structure representation when more complicated expressions must be stored.

Although every representation in a computer must be unambiguous, any given algebraic expression can be represented in a variety of equivalent ways. One might want to construct an algorithm that would always represent an expression in the simplest form possible; in such an algorithm, for instance, $x + x + x$ might always be represented as $3x$. People disagree, however, over what constitutes the simplest form of an expression. Stylistic preferences differ in much the same way as political ideologies do: programs designed by conservatives make no transformations on an expression unless they are specifically instructed by the user to do so, whereas programs designed by radicals always change the user's expression to their own preferred form. Many people like to retain some control over the simplification of an expression because the utility of a particular form may depend on the circumstances in which it is encountered. In any event, even if people could agree on what is the simplest form of every expression, it is known that the construction of a general algorithm for simplification is impossible.

Computer-algebra systems can simplify expressions that are exceedingly large and complex. Such expressions may incorporate not only the elementary polynomial, trigonometric and logarithmic functions but also the more complex functions that can arise in scientific work. The user of a computer-algebra system can also define his own functions, specify their properties and



GCD(102,131;251,591)

$$\begin{array}{r}
 102,131 \overline{) 251,591} \\
 \underline{204,262} \\
 47,329
 \end{array}$$

R1 = 47,329

FLOW CHART diagrams the operation of the Euclidean algorithm, a rigorous procedure for determining the greatest common divisor (GCD) of two integers. (The greatest

supply the rules of simplification appropriate to them; indeed, the computer can sometimes be employed to find such rules. In most cases little is known in advance about how an expression will simplify. The most practical method is to retrieve from the computer's memory all the kinds of algebraic expression the program can already simplify, supply whatever additional algebraic identities the programmer can think of and allow the computer to proceed on its own. The method works surprisingly well: it can often simplify as effectively as an expert mathematician.

Because of the need in algebraic calculations for exact rather than approximate answers, a computer-algebra system must be able to manipulate huge numbers with unlimited precision. In most computers numbers are ordinarily represented with a fixed number of significant digits, although the precision of the arithmetic operations can sometimes be doubled or even increased severalfold. For the purposes of computer algebra, however, no limits on precision can be set in advance.

Consider one algorithm that can be employed to simplify a rational polynomial expression (one in which one polynomial is divided by another). The algorithm is a generalization and modification of a method called the Euclidean algorithm for determining the greatest common divisor (GCD) of two numbers, which has been known for 2,200 years. The GCD of two numbers, such as 6 and 8, is found by a fixed sequence of repeated divisions; once the GCD is found, a fraction can be reduced to its lowest terms by dividing both the numerator and the denominator by the GCD. For the fraction $6/8$ the GCD is 2 and the reduced form is $3/4$.

When the Euclidean algorithm is ap-

plied to a polynomial, however, the quotients and remainders in the repeated divisions can rapidly develop terms in which the fractional coefficients are ratios of enormous numbers. To avoid computing with enormous numbers, certain techniques have been devised for reducing the size of the numbers carried along in the computation. For example, a constant numerical factor can be eliminated from a polynomial in the Euclidean algorithm. Rounding any of the coefficients before the calculation is complete, however, would make nonsense of the final answer.

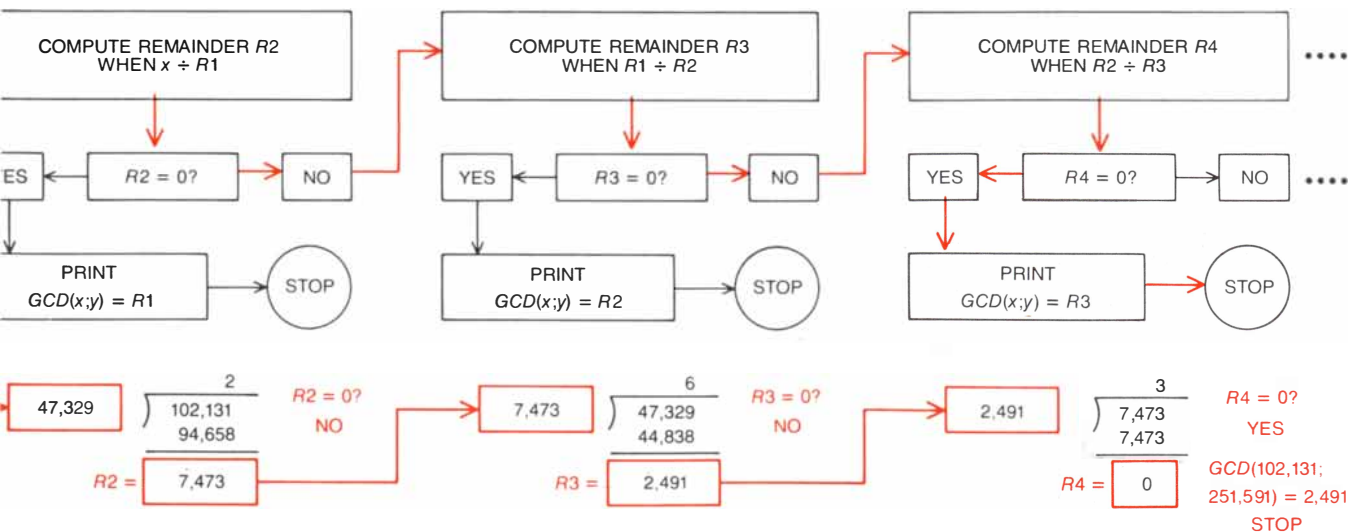
One of the most important algorithms developed in the past 15 years for computer-algebra systems is an algorithm that can solve problems in indefinite integration. Integration is a basic method of calculus, first invented independently by Newton and Leibniz, whereby an unlimited number of arbitrarily small quantities can be combined to yield some specific quantity. In what is called indefinite integration, the method is generalized: the outcome of the integration is expressed as a function of at least one variable instead of as some definite quantity. Such problems are encountered often in the physical and biological sciences, but their difficulty has bedeviled mathematicians for hundreds of years. It was once thought that no general algorithm for the solution of such problems could be constructed; the standard approach to their solution calls both for guesswork and for consulting published tables of integrals.

The algorithm that was successfully constructed for indefinite integration is similar to an algorithm subsequently developed to determine an algebraic expression for a finite sum of expressions that all have the same form [see illustra-

tion on page 139]. If an integral can be solved at all in closed form (rather than as the sum of an infinite series of terms), it is possible to predict the general algebraic form of the solution and then work backward by differentiation (the reverse of integration) to determine the exact formula. The algorithm has been incorporated into several computer-algebra systems. A numerical computer study of eight widely employed tables of indefinite integrals discovered that about 10 percent of the formulas are in error; one of the tables was found to have an error rate of 25 percent.

The funds spent on the development of computer-algebra systems probably do not exceed \$3 million per year in the U.S. Nevertheless, the systems already exhibit great sophistication and diversity of design. There are about 60 such systems today, and they can be classified into three main groups that reflect their historical development. The systems in the first group are the descendants of the earliest attempts to write programs in computer algebra; they were designed to solve specific problems in fields such as mathematical physics and theoretical chemistry. Because a special-purpose program can be finely tuned for the kind of input expected, it can operate at great speed. Currently such programs exist for solving problems in quantum electrodynamics (ASHMEDAI), lunar theory and general relativity (CAMAL), high-energy physics (SCHOONSCHIP), indicial tensor manipulation (SHEEP) and celestial mechanics (TRIGMAN) and for solving equations limited to polynomials and rational functions (ALTRAN). There is even a special-purpose program called ALDESAC/2 that can assist in developing new computer-algebra programs.

The systems in the second major



common divisor is the largest integer that will divide both given integers without leaving a remainder.) The algorithm was discovered at least 2,200 years ago; except for the four elementary operations of arithmetic it is the oldest algorithm known. Colored arrows in the

flow chart indicate the transitional states of the algorithm for the numerical example in the lower part of the diagram. The Euclidean algorithm is not limited to integers; a slightly modified procedure can determine the greatest common divisor of two polynomials.

group are the general-purpose ones, which provide the investigator with as many mathematical capabilities as possible. Such systems can carry out the four basic arithmetic operations, perform definite and indefinite integration and solve equations, including ordinary differential equations that express algebraically how the change in one variable

depends on changes in other variables. The programs can also solve systems of linear or nonlinear algebraic equations, differentiate, simplify, factor, compute with finite or infinite sums of expressions called Taylor series and perform all the functions of the special-purpose systems, although not with as much speed and power. The four best-known

general-purpose systems are MACSYMA; REDUCE, a system developed at Stanford University, the University of Utah and the Rand Corporation; SCRATCHPAD, developed by the International Business Machines Corporation, and SMP, developed at the California Institute of Technology. They represent the most advanced achievements in computer algebra so far.

Computer-algebra systems designed to operate with microcomputers have now begun to appear. They are slower and less comprehensive than the general systems designed for large computers, but they can still perform far more complex calculations more accurately than many mathematicians can. The most sophisticated and widely available such system is called MUMATH; it was developed at the Soft Warehouse in Honolulu. MUMATH provides some of the capabilities of the general systems, although the size, memory and speed of the microcomputer do not allow MUMATH to attack very complex problems. Nevertheless, such systems (or their more powerful descendants) may find their way into personal microcomputers and perhaps even notebook-size calculators before the end of the decade.

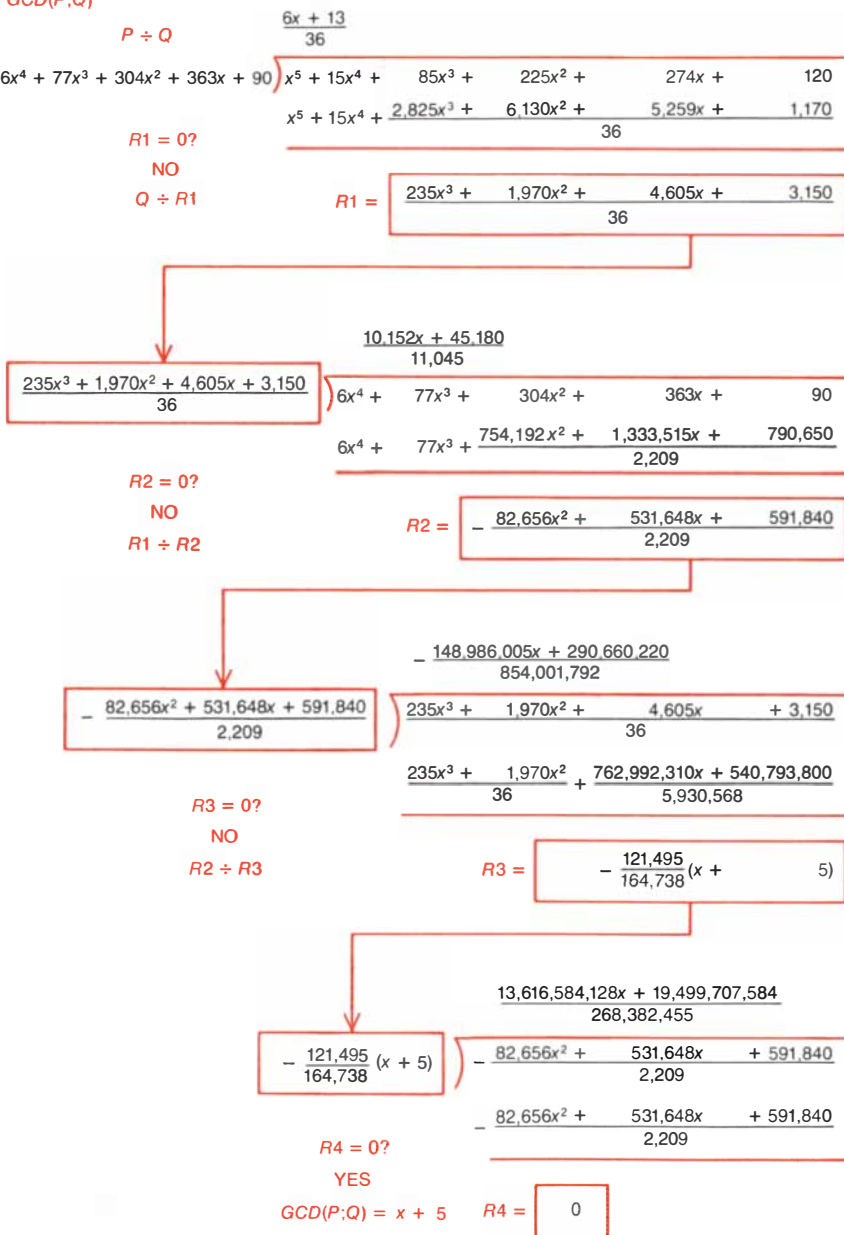
Computer-algebra systems tend increasingly to be written in simple, high-level programming languages that resemble ordinary mathematical expressions. The systems are also generally designed to interact with the programmer in a conversational manner. The beginner can often learn in a few minutes to solve many kinds of nontrivial problem beyond his power to solve by hand. With most general systems the user types in a command in a language similar to the computer languages ALGOL or FORTRAN. If the command includes fractions, exponents or other symbols that do not normally appear on the same line in print, the computer replies with a rewritten version of the expression that displays it in more natural form. The user can then specify certain operations to be carried out on the expression, such as adding it to itself, raising it to a power, differentiating it or integrating it. The computer executes the operation, simplifies the resulting expression and prints it or displays it on a video terminal. If the operation depends on, say, whether the positive or the negative square root of an expression is to be extracted, the computer queries the user before the calculation is finished.

Computer algebra has been applied in a variety of disciplines, including acoustics, algebraic geometry, economics, fluid mechanics, structural mechanics and number theory, and in the design of propellers, ship hulls, helicopter blades, electron microscopes and large-scale integrated circuits. We shall describe three additional applications in which com-

$$P = x^5 + 15x^4 + 85x^3 + 225x^2 + 274x + 120$$

$$Q = 6x^4 + 77x^3 + 304x^2 + 363x + 90$$

GCD(P;Q)



EUCLIDEAN ALGORITHM applied to polynomials determines their greatest common divisor by repeated divisions, in the same way as it determines the greatest common divisor of two integers. The algorithm can give rise to huge intermediate numerical results that cannot be rounded off. For this reason the arithmetic operations in a computer-algebra system must have unlimited precision. Ordinarily a number stored in a computer is allotted a fixed amount of space in the computer memory, but in a computer-algebra system no fixed allocation can be made. Mathematicians who study the properties of large numbers have been attracted to this feature of computer-algebra systems. Colored arrows and the statements printed in color correspond to the steps of the algorithm diagrammed in the illustration on pages 144 and 145.



Olympus OM-1. Every inch a classic. Inside and out.

You are looking at the most revolutionary SLR of all time. The camera more than a *million* photographers have made into a modern classic. The *original* SLR compact, proven and perfected like no other. Elegant. Smooth. Precise. Enduring. That's the Olympus OM-1. A classic.

A classic in control. Its ultra-precise manual meter lets you control exposures with fingertip ease, always. Its finder is free of the clutter so common in conventional cameras—and far larger and brighter as well, thanks

to its precious silver coating, multi-coated mirror, and superb Olympus optics.

A classic in ruggedness. Not for OM-1 are the plastic panels of ordinary SLR's. Here, you'll find a solid, all-metal body. Metal gears born of aerospace technology. Chromed-steel lensmount that defies wobble and wear. And more than a score of vibration-taming shock absorbers. So OM-1 works flawlessly under temperatures and conditions so extreme, others won't work.

A classic in capabilities. The world's first compact 5 shots-per-second motor drive, and the world's most ingenious automatic flash units. Dozens of matchless Olympus lenses, more than 300 matched accessories in all. All fully compatible with every OM-1 ever made. From the largest compact system anywhere.

See OM-1 at your Olympus dealer. Alone in its class, because it's a classic. For information, write Olympus, Woodbury, NY 11797. In Canada: W. Carsen Co. Ltd., Toronto.

OLYMPUS® OM-1



Truck-mounted vibrator plates in contact with the ground send energy waves into the earth, resulting in the seismic mapping of the subsurface.

Watch The Nightly Business Report on your local PBS TV station. Made possible, in part, by a grant from Cities Service Company.

**"We're taking
great color photos
of Oklahoma—
16,000 feet down."**



C.J. Waidelich, President and Chief Executive Officer,
Cities Service Company, Tulsa, Oklahoma

A major breakthrough in seismology is increasing our success ratio in finding new energy reserves.

Mapping underground terrain—like we're doing here in Oklahoma—is nothing new. But Cities Service geophysicists are taking basic seismic data and adding a significant twist—the ability to isolate and recognize geophysical findings and attributes through a unique color process.

We call this development Cit-Chrome.* Now when we look at seismic sections, we're able to see things we might have missed before.

Color seismic is helping us do a better job of finding new deposits—before a wildcat is even drilled. So when we *do* drill, our ratio of successful wells goes up.

This is just one of many advanced technological tools Cities Service is using. We're also using computers that process seismic data so that the subsurface can be visualized in three dimension, and satellite telemetry photos to help pinpoint untapped energy resources.

Cities Service geophysical information from all over the world is fed into our Technology Center in Tulsa for in-depth analysis. It's the newest, and one of the most advanced centers of its kind. This multimillion dollar complex symbolizes the commitment we've made to change America's energy future for the better.

CITIES SERVICE COMPANY

**On top of the problem,
part of the solution.**

*Registered Trademark Cities Service Company

Everything about David Plastow represents his company's philosophy. Which is why he wears a Rolex.

David Plastow is the custodian of a long and famous engineering tradition.

He is the Chief Executive of Rolls-Royce Motors, and indeed, his manner and personal appearance exactly reflect the ethos of that company. That of the skilled engineer.

Plastow takes a personal interest in any modification, however small.

"All our developments at Rolls-Royce are always evolutionary rather than revolutionary," he says.

"We are a highly personal business, and both our craftsmen and our customers have clearly defined ideas about what a Rolls-Royce should be. But while we don't tamper with those fundamental ideas, we are, of course, constantly searching for improvement. For instance, years ago, the gear selection on a Rolls-Royce car became completely electronic. But, a driver likes to 'feel'



that the gear selection lever is doing something . . . so we engineered the 'feel' back into it – so it's satisfying to use."

David Plastow recognizes the similar philosophy behind the watch he wears.

"It's a Rolex Oyster Datejust. I'm told that the engineering concept of the Oyster case first appeared in 1926.

"Obviously this watch has changed and improved over the years but Rolex has stayed with the basic

idea because it was a very good one. It's extremely tough, very reliable, and superbly engineered. After 50 years of development it's almost perfect".

Which, from the man who makes the finest cars in the world, is quite a compliment.



ROLEX



The Rolex Datejust Chronometer. Available in 18kt. gold, steel and gold combination or stainless steel, with matching bracelet.

*Write for brochure. Rolex Watch, U.S.A., Inc., Dept. 298, Rolex Building, 665 Fifth Avenue, New York, N.Y. 10022.
World headquarters in Geneva. Other offices in Canada and major countries around the world.*

puter algebra is making contributions. For investigations in plasma physics and the development of fusion-energy sources some calculations require numerical and algebraic computer techniques to be combined. The algebraic formulas that describe the properties of a plasma in a magnetic field can become quite complex, and they tend to impede physical understanding. What has been lacking is a means for obtaining approximate rather than exact analytic expressions for such phenomena. To solve the problem the Plasma Theory Group at

M.I.T. has devised a computer-algebra technique capable of partitioning the terms in the computation according to the physical processes that generate them. By numerical methods one can then eliminate from the series of terms describing each process those that are relatively small. The result is an algebraic expression that describes the dominant properties and remains sufficiently simple for the physical significance of each term to be perceived. In order to derive testable predictions from theories of the interactions of ele-

234

THÉORIE DU MOUVEMENT DE LA LUNE.

la fonction R ne contient plus aucun terme périodique; elle se trouve donc réduite à son terme non périodique seul, terme qui, en tenant compte des parties fournies par les opérations 129, 260, 349 et 415, a pour valeur

$$R = \frac{\mu}{2n} + m \frac{n^2}{a^3} \left\{ \frac{1}{4} - \frac{3}{2} \gamma^2 + \frac{3}{8} e^2 + \frac{3}{8} e^4 + \frac{3}{2} \gamma^4 - \frac{9}{4} \gamma^2 e^2 - \frac{9}{4} \gamma^2 e^4 + \frac{9}{16} e^2 e^4 + \frac{15}{32} e^6 - \frac{33}{2} \gamma^2 e^2 \right. \\ + \frac{9}{4} \gamma^4 e^2 + \frac{75}{16} \gamma^2 e^4 - \frac{27}{8} \gamma^2 e^2 e^4 - \frac{45}{16} \gamma^2 e^6 + \frac{45}{64} e^8 \\ + \left(\frac{9}{16} \gamma^4 + \frac{225}{64} e^2 - \frac{27}{16} \gamma^2 - \frac{387}{32} \gamma^2 e^2 + \frac{23}{16} \gamma^2 e^4 - \frac{225}{128} \gamma^2 e^6 + \frac{825}{64} e^2 e^4 + \frac{9}{8} \gamma^4 \right. \\ \left. + \frac{3897}{64} \gamma^2 e^4 - \frac{99}{16} \gamma^2 e^2 - \frac{1431}{256} \gamma^2 e^6 - \frac{1419}{32} \gamma^2 e^2 e^4 - \frac{225}{512} e^6 - \frac{825}{128} e^2 e^4 \right) \frac{n^2}{n} \\ - \left(\frac{31}{32} - \frac{33}{8} \gamma^2 - \frac{971}{32} e^2 + \frac{465}{64} e^4 + \frac{273}{64} \gamma^4 + \frac{5709}{64} \gamma^2 e^2 - \frac{117}{4} \gamma^2 e^4 + \frac{4989}{256} e^6 \right. \\ \left. - \frac{1905}{8} e^2 e^4 + \frac{3255}{128} e^6 \right) \frac{n^2}{n^2} \\ - \left(\frac{255}{32} - \frac{31515}{1024} \gamma^2 - \frac{551115}{4096} e^2 + \frac{6885}{64} e^4 + \frac{20511}{512} \gamma^2 + \frac{927831}{2048} \gamma^2 e^2 \right. \\ \left. - \frac{218115}{512} \gamma^2 e^4 + \frac{1622985}{16384} e^6 - \frac{4069635}{2048} e^2 e^4 \right) \frac{n^2}{n^2} \\ - \left(\frac{5515}{192} - \frac{296779}{3072} \gamma^2 - \frac{6380965}{12288} e^2 + \frac{16285}{24} e^4 \right) \frac{n^2}{n^2} \\ - \left(\frac{28841}{288} - \frac{113818307}{294912} \gamma^2 - \frac{1681901031}{1179648} e^2 + \frac{1293609}{384} e^4 \right) \frac{n^2}{n^2} \\ - \frac{9814775}{36864} \frac{n^4}{n^2} - \frac{428268199}{663552} \frac{n^4}{n^2} \\ + \left[\frac{9}{64} - \frac{45}{16} \gamma^2 + \frac{45}{64} e^2 + \frac{15}{128} e^4 \right. \\ \left. + \left(\frac{225}{512} - \frac{1935}{256} \gamma^2 + \frac{7425}{1024} e^2 + \frac{225}{64} e^4 \right) \frac{n^2}{n} + \frac{869}{512} \frac{n^2}{n^2} - \frac{10391}{8192} \frac{n^2}{n^2} \right] \frac{n^2}{n^2} \Big\}$$

MASSIVE ALGEBRAIC CALCULATION completed by hand in 1867 was recomputed for the first time in 1970 with a computer-algebra system. The original calculation was undertaken by the French astronomer Charles Delaunay and was published in two volumes; a page of the second volume is reproduced here. Delaunay's calculation, which took 10 years to finish and another 10 years to check, gave the position of the moon as a function of time to a precision never before attained. Three errors were detected when the calculation was checked by André Deprit, Jacques Henrard and Arnold Rom of the Boeing Scientific Research Laboratories in Seattle. The major error, which gave rise to the other two errors, is outlined in color; the expression should be $33/16 \gamma^2 e^2$. The checking required about 20 hours of running time on a computer. Although computer algebra was responsible for exposing Delaunay's errors, the tables have since been turned: the remarkable precision of the calculation has been employed to test a few new computer-algebra systems for accuracy before they were put into service.

No more cold coffee



with MUG-MATE
Your personal cup heater.

Keep your coffee/tea/soup hot to the last drop on this miniature hot plate. Perfect for office, home or school! Money back guarantee. **Special Christmas Gift Offer!** 10% discount on orders of 12 or more. Send us your gift list; we'll ship with gift card in your name. Christmas delivery guaranteed.

RUSH me _____ MUG-MATE s @ \$12.95 + \$1.50 shipping each (in Calif. add 78¢ S.T.). Enclosed is check/M.O. or charge to Visa or MasterCard. Card # _____ Exp. _____

Name _____
Address _____ Apt _____
City _____ State _____ Zip _____
HONEX
1485 Bayshore Blvd., San Francisco, CA 94124

Garden Camera CALCULATORS

Hewlett Packard

HP 67	\$288.50
HP 97	568.50
HP 31E	34.50
HP 32E	42.50
HP 37E	57.95
HP 33C	68.50
HP 34C	114.50
HP 38C	114.50
HP 41C	188.50
HP 41CV	Call
System I	Call
Accessories	Call

Texas Instruments

TI 58C	79.95
TI 59	188.50
PC 100C	155.00
TI M8A	52.50

Chess Challenger

Level 7	78.50
Level 8	104.95
Level 10	119.50
Voice	229.50
Bridge	229.50

Prices subject to change without notice. Speed your order. TOLL FREE!

Call 1 (800) 223-0595
Or Send postage and handling to
GARDEN CAMERA
345 Seventh Avenue, N.Y., N.Y. 10001
New York, Alaska & Hawaii Call:
Tel: (212) 868-1420 Open Weekdays 8:30-6:00
OPEN SUNDAYS 10-4 p.m. Closed Saturdays

apple computer



BEST PRICES U.S.A.

- APPLE II II PLUS (16K, 32K, 48K)
- APPLE III
- DOS 3.3
- APPLE PLOT
- APPLE PASCAL (64K)
- APPLE II OR I RAM
- MSICAL C 16 SEC
- VISIPLLOT
- VISITREND
- APPLE WRITER
- GRAPHICS TABLET
- GPI (GL. AR. & PAYROLL)
- MDDM
- DOW JONES NEWS & QUOTES
- DECISION EVALUATOR
- CONTROLLER
- MICRO-COURIER
- EPSON
- CENTRONICS
- QUME
- SILENTTYPE
- ANADIX
- SANYO. BW. COLOR
- GREENSCREEN
- EXTENDED WARRANTY

GET A \$20 REBATE ON THE TI-59 PROGRAMMABLE.

Texas Instruments

TI-59 960 PROG	179.95		
PC-100C	169.95		
LCD-PROG NEW	59.95		
TI-30II NEW	18.95		
TI-35SP SCI	22.50		
TI-40 SCI NEW	28.95		
BUS ANAL I	19.95	SPEAK & SPELL READ	59.95
BUS ANAL II	44.95	SPEAK & MATH	59.95
BUS CARD	39.95	TOUCH & TELL NEW	54.95
MBA	54.95	TI-5100 DISPLAY	39.95
INVEST ANALYST	48.95	TI-5100 HAND/PRINT	49.95
TI-54 SCI NEW	39.95	TI-5120 PRINTER	59.95
TI-55II NEW	44.95	TI-5130 PRINT/DISP	79.95
TI-57 PROG SCI	39.95	TI-5135 PRINT/DISP	79.95
TI-58C PROG CALC	89.95	TI-5142 PRINT/DISP	89.95



Your hp HEWLETT PACKARD Headquarters

THE HP-85!

Complete Enhancements, Peripherals and Accessories



HP-67	289.95	HP-37E BUS	59.95
HP-97	584.95	HP-11C NEW	CALL
HP-33C SCI	79.95	HP-12C NEW	CALL
HP-34C SCI	114.95	HP-43, 41CV	CALL
HP-38C BUS/RE	119.95	HP-85	CALL
HP-32E SCI	49.95	HP-125 NEW	CALL



TOUCH THE FUTURE

ATARI 400 (16K)	349.95
VISICALC AVAILABLE	CALL



CHESS CHALLENGER 7	89.95
SENSORY CHESS	129.95



5813 SCI PROGRAMMABLE	34.95
1182A PRINT/DISPLAY	74.95
TALKING CLOCK	79.95
EL-6200 DIG EXEC SEC	89.95



AA-81 DIG/ANALOG ALARM	69.95
VL-TONE MUSICAL INSTRUMENT/CALC	69.95
W100 DEPTH TESTED ALARM CHRONO	39.95
FX7100 SCI CHRONO ALARM CALC	49.95
FX3500 SCI PROGRAMMABLE CALC	39.95

(714) 549-7373 INFORMATION LINE

(800) 432-7066 TOLL FREE (within CA)

(800) 854-0523 TOLL FREE (Outside CA)

WE WILL MEET OR BEAT ANY COMPETITOR'S ADVERTISED PRICE ON MOST ITEMS IF HE HAS THE MDSE. ON HAND. VISA, MASTERCARD, MONEY ORDER, PERS. CK. (14 WRKG. DAYS TO CLR.), COD ACCEPTED; MIN. \$4.95 SHIPPING U.S.A.; AIR ON REOST; CAL RES. ADD 6% SALES TX. ALL MDSE. SUBJ. TO AVAIL.; PRICES SUBJ. TO CHANGE. S-A-D

MAIL & PHONE ORDERS ONLY

WRITE OR CALL FOR FREE CATALOG

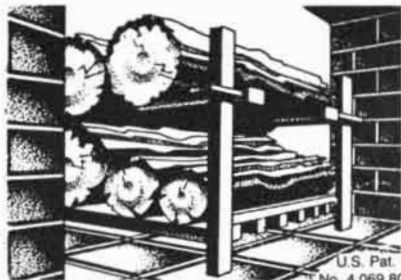


3211 SO. HARBOR BLVD. SANTA ANA, CA 92704 NEWPORT (714) 549-7373

PASADENA (213) 795-3007 MID-WILSHIRE (213) 385-7777
 TARZANA (213) 705-7507 LAWDALE (213) 370-5795
 WEST LOS ANGELES (213) 8200423 BREA (714) 990-6600

PROFESSIONAL DISCOUNTS

"Physicist's Fire" with the Texas Fireframe® Grate



"Amazing amount of heat" BETTER HOMES AND GARDENS; "Easy to start" TIME; "slow-burning" NEW YORK TIMES; "No rotation or stirring of the logs" SCIENTIFIC AMERICAN; "2.6 x more efficient" POPULAR SCIENCE. "Easy to maintain" D. J. Ticko, New Fairfield, CT. "Does a fantastic job" Frank Stanton NYC.

For full scientific description, see L. Cranberg, Am. Jour. Physics, June '81. Reprints on request. Grate Model S-25: 25" front width, 21" back width, 13" deep. Model KS-25: same, heavy-duty, gift-boxed. Model U-25: 25x21x15. Model U-17: 17x14x13. Model U-33: 33x29x15. Copyrighted instructions.

- ___S-25 @ \$44.95 (26 #); ___U-25 @ \$44.95 (28 #)
- ___U-17 @ \$36.95 (20 #); ___U-33 @ \$56.95 (35 #)
- ___KS-25 @ \$51.95 (31 #); ___Reprint (s-a envelope)

Add 10% for shipping in U.S.; Enclose check or Visa, MC # _____ Exp. Date _____

Name _____

Address _____

City _____ State _____ Zip _____

TEXAS FIREFRAME CO. P.O. Box 3435 Austin, Texas 78764

Anyone for Tenons?



If your racket is tenons, doubles or singles, and you want a perfect match every time, you can learn from the professionals. They know a classic way to join two pieces of wood together is with a tenon that fits into a mortise.

Fine Woodworking magazine speaks this language to more than 200,000 readers—people seriously interested in making beautiful things out of wood. Whether you are an aspiring novice or an accomplished expert, you'll find it an especially informative magazine, with finely detailed articles about techniques and tools, materials and design. It's written by master craftsmen who share their many years of woodworking experience. And it's printed on heavy paper to withstand years of reference. All back issues are in print and indexed periodically.

If you want to become a better woodworker, send \$14.00 for a one-year subscription (6 issues). Or call us toll-free, 1-800-243-7252, to charge your order to Visa or MasterCard (Connecticut residents call 1-426-8171).

Fine Woodworking®

The Taunton Press 52 Church Hill Rd. P.O. Box 355KC Newtown Connecticut 06470

mentary particles it has been necessary to evaluate expressions that can include thousands of integrals, many of which have unusual and deceptive mathematical properties. The numerical evaluation is open to two of the criticisms of numerical methods we have already mentioned: it is not as useful as an algebraic result for showing how contributions from various physical phenomena affect the interaction, and it introduces errors of approximation. As experimental techniques are refined the uncertainties introduced by approximation can become so large that the predictions of the theories cannot be distinguished. It is likely the application of computer-algebra methods will be of great help in generating theoretical predictions of the required precision.

In the study of theoretical alternatives to general relativity a test called Birkhoff's theorem has recently become popular. The mathematician George David Birkhoff of Harvard University showed in 1923 that the general theory of relativity excludes the propagation through space of gravitational pulses that might conceivably be generated by radial pulsations of matter in a star. Because no such pulses have been detected, and because they are impossible according to both Einstein's and Newton's theories of gravity, the exclusion of the pulses has come to be expected of any potential gravitational theory. The calculation required to determine whether or not a theory of gravity satisfies Birkhoff's theorem is lengthy, but it has now become straightforward with computer-algebra systems. A gravitational theory formulated by C. N. Yang of the State University of New York at Stony Brook was recently shown by the use of a computer-algebra system to violate Birkhoff's theorem. The calculation corrected a hand calculation that had been believed and cited for many years.

It is important to recognize that computer-algebra systems need not be employed solely for large calculations. Relatively simple problems often arise for which algebraic manipulation can contribute to understanding or make numerical calculation more efficient. One can sometimes reduce numerical computer-processing time by a factor of 100 or more by the judicious application of algebraic processing. Moreover, computer-algebra programs can substitute for an entire library of mathematical reference works. "Knowledge-based" programs that include integration methods, methods for solving differential equations and the like may make such reference works as obsolete as a table of logarithms. As inexpensive computers are improved, computer algebra will become available for teaching, study, research and perhaps unthought-of applications to all interested people in their own offices and homes.

explore the



world below

Oceanus[®]

Published by
Woods Hole Oceanographic Institution
one of the world's leading centers of ocean research

Marine Scientists talking
directly to the serious student
of the sea

SUBSCRIPTION DEPT. BOX H
1172 COMMONWEALTH AVENUE
BOSTON, MASSACHUSETTS 02134

Please make checks payable to Woods Hole Oceanographic Institution,
Woods Hole, Massachusetts 02543.

Please enter my subscription to OCEANUS for one year at \$15.00 (four issues).
 payment enclosed bill me

Name.....Address

City.....State.....Zip