Approfondimenti

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1 Laboratorio di Acquisizione e Controllo Dati

1.1 Lezione #1

1.1.1 Telegrapher's Equations

Some definitions first:

$$\epsilon_0 = \text{ vacuum permittivity } = 8.8544 \times 10^{-12} \text{ F/m}$$

$$\mu_0 = \text{ vacuum permeability } = 1.5266 \times 10^{-6} \text{ H/m}$$

$$1 \text{ F (Farad) } = \frac{s^4 A^2}{kg\ m^2} \quad ; \quad 1 \text{ H (Henry) } = \frac{m^2 kg}{A^2 s^2}$$

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \sim 3.0 \times 10^8 \text{ m/s}$$

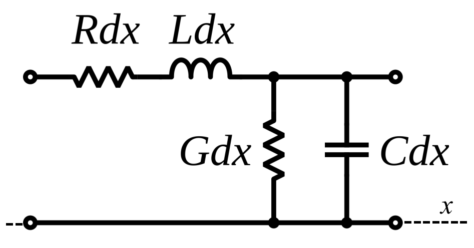

Figura 1: two parallel wires

Let's now assume we have two infinite length wires running parallel to each other as in fig. 1, (courtesy of *Wikipedia*) and for simplicity we consider the system uni-dimensional (along x). Let us indicate R, C, G and L, the resistance (Ohm), capacitance (Faraday), conductance (Siemens) and inductance (Henry), respectively, and we assume that these values are kept constant along the wire. From Electromagnetism they are defined as:

$$R = \frac{V(x,t)}{I(x,t)} \quad ; \quad C = \frac{Q(x,t)}{V(x,t)}$$

$$G = \frac{I(x,t)}{V(x,t)} \quad ; \quad L = \frac{\Phi_B(x,t)}{I(x,t)}$$

where $V(x,t)$, $I(x,t)$, $Q(x,t)$ and $\Phi_B(x,t)$ are the Voltage, Current, Charge and Magnetic Flux, respectively, and we know the relations between them:

$$V(x,t) = \frac{\partial \Phi_B(x,t)}{\partial t} = L\frac{\partial I(x,t)}{\partial t}$$

$$I(x,t) = \frac{\partial Q(x,t)}{\partial t} = C\frac{\partial V(x,t)}{\partial t}$$

where, as said before, we have assumed that these quantities are only function of x (uni-dimensional) and t. Let Δx be the distance between two points along the wire where we measure V and I. Also let us substitute R, G, C and L by the same quantities *per unit length.* Then the voltage drop will be given by two quantities:

$$\Delta V(x,t) = -R\Delta x I(x,t) - L\Delta x\frac{\partial I(x,t)}{\partial t} \tag{1}$$

due to R and L along the wire (series) and, similarly, the Current drop after the same distance:

$$\Delta I(x,t) = -G\Delta x V(x,t) - C\Delta x\frac{\partial V(x,t)}{\partial t} \tag{2}$$

due to the C and G between the wires (parallel). The negative signs are due to the fact that V and I decrease while moving along the positive x axis If we now divide both equations 1 and 2 by Δx and take the limit $\Delta x \to 0$ (and removing "(x,t)" for simplicity) we obtain the two **Telegrapher's equations**

$$\begin{cases} \dfrac{\partial V}{\partial x} = -RI - L\dfrac{\partial I}{\partial t} \\ \dfrac{\partial I}{\partial x} = -GV - C\dfrac{\partial V}{\partial t} \end{cases}$$

We now derive the first equation w.r.t. x and the second equation w.r.t. t and obtain

$$\begin{cases} \dfrac{\partial^2 V}{\partial x^2} = -R\dfrac{\partial I}{\partial x} - L\dfrac{\partial^2 I}{\partial t\partial x} \\ \dfrac{\partial^2 I}{\partial x\partial t} = -G\dfrac{\partial V}{\partial t} - C\dfrac{\partial^2 V}{\partial t^2} \end{cases}$$

then substitute $\partial^2 I/\partial x\partial t$ in the first equation and combine the two equations into a single one

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2} + (GL+RC)\frac{\partial V}{\partial t} + RGV \tag{3}$$

We can find a similar equation with Current in place of Voltage:

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2} + (GL+RC)\frac{\partial I}{\partial t} + RGI$$

Lossless transmission line

The ideal situation is that in which $R = 0$ (the wires have no resistance: perfect conductors) and $G = 0$ (no conductance between the wires: perfect insulation), then eq 3 becomes

$$\frac{\partial^2 V}{\partial x^2} = LC \frac{\partial^2 V}{\partial t^2} \tag{4}$$

This is the famous wave equation already seen long time ago (first year university: mechanical waves...) and we already know that the solution of this equation are two waves moving in the opposite directions

$$V(x,t) = V_1 e^{i(\omega t - kx)} + V_2 e^{i(\omega t + kx)}$$

with the velocity of propagation of the signal given by

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{LC}}$$

Coaxial cables

A flexible coaxial cable, such as the one depicted in fig. 2 (courtesy of *Wikipedia*) is composed of four coaxial elements

1. outer plastic sheath
2. woven copper shield
3. inner dielectric insulator
4. copper core

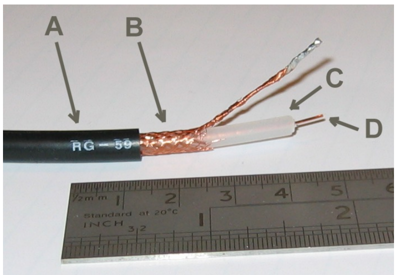

Figura 2: Coaxial cable RG-59

therefore a coaxial cable is like two parallel wires, separated by an insulator. From electromagnetism we know that

$$C = 2\pi\epsilon \left(\ln\frac{b}{a}\right)^{-1} \sim 55.6\epsilon_r \left(\ln\frac{b}{a}\right)^{-1} \qquad \text{[pF/m]}$$

and

$$L = \frac{\mu}{2\pi}\ln\frac{b}{a} \sim 0.2\mu_r \ln\frac{b}{a} \qquad [\mu\text{H/m}]$$

where a e b are the diameters of the copper core and the copper shield, respectively, and

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} \quad \text{and} \quad \mu_r = \frac{\mu}{\mu_0}$$

with ϵ_r the relative permittivity and μ_r the relative permeability of the dielectric insulator. From the above information, we can find the speed of the wave inside a coaxial cable, and it depends only on the characteristic of the inner dielectric insulator

$$LC = \epsilon\mu \rightarrow v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\epsilon\mu}} = \frac{1}{\sqrt{\epsilon_r\epsilon_0\mu_r\mu_0}} = \frac{c}{\sqrt{\epsilon_r\mu_r}}$$

From the general telegraph's equations we can find the solutions of the equations, that is $V(x,t)$ and $I(x,t)$ and from that, we can evaluate the **impedance** of the cable which is

$$Z = \frac{V}{I} = \sqrt{\frac{R + i\omega L}{G + i\omega C}}$$

and in the lossless case it becomes

$$Z = \sqrt{\frac{L}{C}} = \sqrt{\frac{\mu}{4\pi^2\epsilon}} \ln\frac{b}{a} \sim 60 \ln\frac{b}{a}\sqrt{\frac{\mu_r}{\epsilon_r}}\ \Omega\ \text{(Ohm)}$$

Therefore the impedance of the coaxial cable depends on the dielectric insulator and on the ratio of the diameters of the copper shield and core.

Real case transmission line

A true coaxial cable is not an ideal lossless cable and therefore we must take into account the fact that the telegraph's equation cannot be simplified to equation 4. We must therefore find a solution to equation 3. Let us assume that we have a signal of the form $V(x,t) = V(x)e^{i\omega t}$. Substituting into eq. 3 one finds

$$\frac{\partial^2 V}{\partial x^2} = (R + i\omega L)(G + i\omega C)V = \gamma^2 V$$

with

$$\gamma = \alpha + ik = [(R + i\omega L)(G + i\omega C)]^{1/2}$$

The equation still has a solution of the kind seen before (two waves moving in opposite directions), but this time with an attenuation term in it

$$V(x,t) = V_1 e^{-\alpha x} e^{i(\omega t - kx)} + V_2 e^{\alpha x} e^{i(\omega t + kx)}$$

The solution is more complicated than this, since $\alpha = \alpha(\omega)$ and therefore the attenuation is not the same at all frequencies. A good approximation is

$$\alpha \sim \frac{1}{2}\left(R\sqrt{\frac{C}{L}} + G\sqrt{\frac{L}{C}}\right)$$

but it is seen that it varies even tenfold at high frequencies. The behavior of R, C, G, and L in function of the frequency and their approximate values at 1 MHz (in parenthesis) are:

- $R \propto \sqrt{\omega}$ (500 Ω/km)
- C is practically constant over the whole frequency range (52 nF/km)

- $G \propto \omega$ (30 μ S/km)
- L reduces very little with increasing frequency, some 20% from 10 kHz to 5 MHz (0.5 mH/km)

The attenuation is defined in dB/m.

Some numbers

We use two tipe of cables in our experimental laboratory

Cable	a (mm)	b (mm)	b/a	β	ω_r	ϵ_r	Z (Ω)	α (dB/m)
RG-58C/U (BNC)	2.95	5.00	1.70	0.66	1	2.3	50	0.1
RG-174/U (Lemo)	1.50	2.55	1.70	0.66	1	2.3	50	0.2

Therefore the speed of the signal is approximately 2/3 of the speed of light. That is why the length of the cables is measured in *ns* and not in *cm*. Given the amplitude of a generated signal, its attenuation becomes relevant after a certain distance of the cable. The attenuation given in the table is a rough approximation, because it depends on the frequency, for instance for an RG-58C/U cable the attenuation is $\alpha = 0.06$ and 0.17 at 10 and 100 MHz, respectively.