Mutual inductance

Consider two nearby wire loops or circuits. We define the flux of the field $B$ produced by the circuit (1) through the second circuit (2):

$$\Phi_{2,1} = \int_{S_2} \vec{B}_1 \cdot \hat{u}_n \, dS$$

where $S_2$ is a generic surface that has as contour the circuit (2). Since $\Phi_{2,1}$ is prop to $i_1$

$$\Rightarrow \Phi_{2,1} = M_{2,1} i_1$$
Let's do the same exchanging \((1) \leftrightarrow (2)\)

\[
\Phi_{1,2} = \int_{S_1} \vec{B}_2 \cdot \hat{u}_n \, dS
\]

where \(S_1\) is a generic surface that has as contour (1).

Since \(\Phi_{1,2}\) is prop to \(i_2\) \(\Rightarrow\) \(\Phi_{1,2} = M_{1,2} i_2\)

It can be shown* that:

\[
M_{1,2} = \frac{\Phi_{1,2}}{i_2} = \frac{\Phi_{2,1}}{i_1} = M_{2,1}
\]

*(soon we will demonstrate it)
We can define as mutual inductance coefficient $M$ between two circuits:

$$\Phi_{2,1} = M i_1, \quad \Phi_{1,2} = M i_2$$

Expressing the concatenated magnetic flux due to the magnetic field generated by the current in the other circuit.

Two circuits for which $M \neq 0$ are called coupled circuits. They will be completely characterized only knowing by their resistance, by their inductance and by their coefficient $M$. In Italian “coefficiente di mutua induzione” o di “mutua induttanza”.
As \( L, M \) depends on the shapes of the two circuits, by their relative positions in space and by the magnetic properties of the material(s) in the space around (described by the magnetic permeability coefficient \( \mu, \mu_0 \) if vacuum):

\( M \) depends only by **GEOMETRY AND \( \mu \).**

The unit for \( M \) is the same as the one for \( L \) (H, **henry**), since its definition is very similar, proportional coefficient between a magnetic flux and a current.
Esempio 10.11: Spira singola all’interno di un solenoide

Dati a e n calcolare:

a) M data la geometria
b) M per n=2200 sp/m a=12mm
c) fem spira per 1.4 A/s
d) Verso corrente indotta

Figura 10.10
Esempio 10.11: Una corrente viene indotta nella spira a causa di una variazione della corrente che percorre il solenoide.
Esempio 8.7 Mazzoldi: Mutual inductance between a long solenoid and a coil

Let's get a long solenoid with section $S_1$, and $n_1$
And a coil with section $S_2$, and $N_2$

The magnetic field generated by the solenoid is $B_1 = \mu_0 n_1 i_1$
the flux through one 1 loop of the coil 2 $\Phi^{(1)} = \mu_0 n_1 i_1 S_1$ (outside the solenoid $B_1 = 0$)
The flux concatenated with the $N_2$ loops of the coil 2:
$\Phi_{2,1} = N_2 \Phi^{(1)} = (\mu_0 n_1 N_2 S_1) i_1$

The mutual inductance coeff. is thus $M = \frac{\Phi_{2,1}}{i_1} = \mu_0 n_1 N_2 S_1$
Mutual induction emf

According to Faraday's law we must have an emf induced in each circuit by the current change in the other:

\[ \mathcal{E}^\text{mutual}_1 = \mathcal{E}_{1,2} = -\frac{d\Phi_{1,2}}{dt} = -M \frac{di_2}{dt} \]

\[ \mathcal{E}^\text{mutual}_2 = \mathcal{E}_{2,1} = -\frac{d\Phi_{2,1}}{dt} = -M \frac{di_1}{dt} \]

This mutual influence has an important consequence in terms of energy. Actually this allows us to prove the 
\[ M_{12} = M_{21} \]
Magnetic energy for coupled circuits (in a fixed position).

Suppose that initially $i_1=0$ and $i_2=0$. We switch on one at a time, first (1) then (2). The first generator spends the energy $U_1=L_1i_1^2/2$ to rise the current from 0 to $i_1$. Subsequently the second generator spends the energy $U_2=L_2i_2^2/2$ to rise the current from 0 to $i_2$, but, at the same time, the first has to work against the mutual emf generated by the second (due to the variation of $i_2$), so it must spend also

$$U_{1,2} = \int fem_1' i_1 dt = \int M_{1,2} \frac{di_2}{dt} i_1 dt = M_{1,2} i_1 i_2$$
Repeating the same argument by changing the switching order of circuits we demonstrate the foregoing, that is

\[ M_{1,2} = M_{2,1} = M \]

And we get also an expression of the magnetic energy of two coupled circuits

\[ U_m = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 + M i_1 i_2 \]
Magnetic energy for coupled and not fixed circuits

What happens if I move the circuits? EG If bring them from far far away to the position in figure?

\[ U_m = \frac{1}{2} L_1 i_1^2 + \frac{1}{2} L_2 i_2^2 + M i_1 i_2 \]

\[ M_i = 0 \Rightarrow U_f > U_i \quad \Delta U_m = M i_1 i_2 \]

\[ dU_m = i_1 i_2 dM \]

Who provides this energy?

\[ dU_{\text{gen}} = 2 i_1 i_2 dM = 2 dU_m \]

the other half is mechanical work \( dU_{\text{mecc}} = dU_m \)